JOURNAL OF FUNCTIONAL ANALYSIS 19, 390–421 (1975)

The Taylor Spectrum and Spectral Decompositions

ŞTEFAN FRUNZĂ

Faculty of Mathematics, University of Jassy, Romania

Communicated by the Editors

Received July 31, 1974

INTRODUCTION

The purpose of this paper is to initiate a study of spectral decompositions for commuting n-tuples of linear continuous operators on Banach spaces, using the Taylor spectrum and his recent results concerning the functional analytic calculus [9, 10].

Our program is rather similar to that of [3], where an axiomatic theory of spectral decompositions for one operator is developed (see also [2, 7]).

We shall begin (Sect. 1) with the definition and the properties of the local spectrum; the idea of the definition is to "restrict" the integrand in the Cauchy–Weil formula [10, Sect. 3]; for this we need to impose a suitable cohomological restriction which, in the case $n = 1$, means the well-known single-valued extension property [5, 3].

In Section 2 we obtain a few cohomological results essential to the following section; perhaps they are also interesting by themselves.

In Section 3 we discuss the decomposable n-tuples; they have a good cohomological behavior (Theorem 3.1), allowing us to prove the uniqueness of the spectral capacity (Theorem 3.2). In the last part of this section we study the behavior of the decomposable n-tuples with respect to the functional analytic calculus.

In the final section (Sect. 4) we introduce the commutator and the quasinilpotent equivalence of two n-tuples and we obtain results similar to those of [3].

PRELIMINARIES

Let X be a complex Banach space which will be kept fixed unless stated otherwise. For an arbitrary open set $U \subset \mathbb{C}^n$, we shall denote by $\mathscr{U}(U, X)$ the space of all X-valued analytic functions on U, and by $\mathscr{B}(U, X)$ the space of all X-valued continuous functions on U, which are infinitely differentiable with respect to $\bar{z}_1, \ldots, \bar{z}_n$ (defined by means of distributions in [10, Sect. 2]); $\mathscr{B}_0(U, X)$ will be the subspace of $\mathscr{B}(U, X)$ consisting of all functions with compact support.

If $\sigma = (s_1, \ldots, s_n)$ denotes an n-tuple of indeterminates and Y is one of the spaces X, $\mathscr{U}(U, X)$, $\mathscr{B}(U, X)$, or $\mathscr{B}_0(U, X)$, we shall denote by $\Lambda^p[\sigma, Y]$ the set of all exterior forms of degree p in s, having coefficients in Y.

The operators will be denoted (following [9, 10]) by small letters ($a, b, \alpha, \ldots$) instead of capital letters as is usual.

Let $a = (a_1, \ldots, a_n)$ be a commuting n-tuple of linear continuous operators on X. The complement in $\mathbb{C}^n$ of the Taylor spectrum, $sp(a, X)$, is the resolvent set of a on X and will be denoted by $r(a, X)$.

The operator on the forms in s having coefficients in X, defined as the left exterior multiplication by $(z_1 - a_1)s_1 + \cdots + (z_n - a_n)s_n$ will be denoted $\alpha(z)$; if the coefficients of the forms are the functions on U, we shall denote by α the operator $(\alpha\psi)(z) = \alpha(z) \wedge \psi(z)$, $z \in U$, and by $\alpha \oplus \bar{\partial}$, the operator

$$[(\alpha \oplus \bar{\partial})\,\psi](z) = [(z_1 - a_1)\,s_1 + \cdots + (z_n - a_n)\,s_n + (\partial/\partial\bar{z}_1)\,d\bar{z}_1 + \cdots + (\partial/\partial\bar{z}_n)\,d\bar{z}_n] \wedge \psi(z), \qquad z \in U$$

(the $2n$-tuple $(s_1, \ldots, s_n, d\bar{z}_1, \ldots, d\bar{z}_n)$ will be written as $\sigma \cup d\bar{z}$).

For what follows it is also necessary to give some explanations about the definition of the Cauchy–Weil integral. If U is an open set containing the spectrum of a and $f \in \mathscr{U}(U, X)$, then the Cauchy–Weil integral of f with respect to a, denoted by $\int_U R_{z-a} f(z) \wedge dz_1 \wedge \cdots \wedge dz_n$ is an element of X, obtained as follows. If we consider the form $sf = fs_1 \wedge \cdots \wedge s_n$ as an element in $\Lambda^n[\sigma \cup d\bar{z}, \mathscr{B}(U, X)]$, then there exists a form $\chi \in \Lambda^n[\sigma \cup d\bar{z}, \mathscr{B}_0(U, X)]$ having the same cohomology class as sf with respect to $\alpha \oplus \bar{\partial}$ (i.e., $sf - \chi = (\alpha \oplus \bar{\partial})\psi$); this is because the application i^* defined in [10, Sect. 1] is an isomorphism (see also [10, Sect. 3]). Next, keep the part of χ containing only $d\bar{z}$ (denoted in [10, Sect. 1] by $\pi\chi$); this is again a form with compact support of degree n in $d\bar{z}$. The Cauchy–Weil integral is then

$$\int_U R_{z-a} f(z) \wedge dz_1 \wedge \cdots \wedge dz_n = \int_U (-1)^n\, \pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n. \quad (0.1)$$

It only depends on the cohomology class of χ.

We shall justify the existence of the form χ in another way, more transparent for our purposes than that given in [10, Sects. 1, 3]. Observe first that, by [10, Theorem 2.16 and Lemma 1.3], we have $H^i(\mathscr{B}(V, X), \alpha \oplus \bar{\partial}) = 0$, $0 \leqslant i \leqslant 2n$, for any open set $V \subset r(a, X)$; thus, taking into account that $(\alpha \oplus \bar{\partial})\, sf = 0$ and applying this remark to $V = U \backslash sp(a, X)$, we deduce that there exists a form $\varphi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathscr{B}(V, X)]$ such that $sf = (\alpha \oplus \bar{\partial})\varphi$; now we have only to multiply φ by a suitable C^∞ scalar function (equal to 0 in a neighborhood of $sp(a, X)$ and to 1 outside of a relatively compact neighborhood of the same set) to obtain a form χ.

Using the Cauchy–Weil integral one may define the functional analytic calculus [10, Sect. 4, Definition 4.2], namely, if f is a scalar analytic function on a neighborhood U of the spectrum of a, then $f(a)$ is given by

$$f(a)\, x = \frac{1}{(2\pi i)^n} \int_U R_{z-a} f(z)\, x \wedge dz_1 \wedge \cdots \wedge dz_n\,, \qquad x \in X. \tag{0.2}$$

In particular, since $1(a) = id$ [10, Sect. 4, Theorem 4.3], we have

$$x = \frac{1}{(2\pi i)^n} \int_U R_{z-a} x \wedge dz_1 \wedge \cdots \wedge dz_n\,, \qquad x \in X. \tag{0.3}$$

1. The Local Spectrum

In the axiomatic theory of spectral decompositions, initiated by Dunford [5, 6], the spectrum relative to an element (so-called local spectrum) is of the greatest importance; for instance, it is essential for obtaining the uniqueness of the spectral measure [5, Theorem 6].

The different succesive generalizations of the Dunford spectral operators have been included in a unitary theory in [3]. This is similar to the Dunford theory (see also [7]) and strongly depends on the local spectrum. In this section, we intend to generalize this notion to commuting n-tuples. Therefore, we want to define the spectrum of an arbitrary element $x \in X$, with respect to a commuting n-tuple $a = (a_1, \ldots, a_n)$ of linear continuous operators on X. To do it, we shall "restrict" the integrand in the formula (0.3).

Consider all open sets V in $\mathbb{C}^n$ such that there exists a form $\psi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathscr{B}(V, X)]$, satisfying $sx = (\alpha \oplus \bar{\partial})\psi$; for instance, the resolvent set, $r(a, X)$, has this property (see the preliminary discussion on the Cauchy–Weil integral).

DEFINITION 1.1. The resolvent set $r(a, x)$ of x with respect to a is the union of all open sets V with the preceding property. The complement of this set (in $\mathbb{C}^n$) is called the spectrum of x with respect to a, $sp(a, x) = \mathbb{C}^n \backslash r(a, x)$.

Remark 1.1. For $n = 1$, the form ψ is of the degree zero, hence it is a simple function in $\mathscr{B}(V, X)$; in this case the equality $sx = (\alpha \oplus \bar{\partial})\psi$ becomes $sx \equiv (z - a)\, \psi(z)s + (\partial\psi/\partial\bar{z})\, d\bar{z}$, $z \in V$, which means $x \equiv (z - a)\, \psi(z)$, and $(\partial\psi/\partial\bar{z}) \equiv 0$; therefore ψ is an analytic function and our definition coincides with that of [3, Chap. 1, Sect. 1].

It is easy to verify the following properties.

1. $x = 0$ implies $sp(a, x) = \varnothing$.
2. $sp(a, x + y) \subseteq sp(a, x) \cup sp(a, y)$ for any $x, y \in X$.
3. $sp(a, bx) \subseteq sp(a, x)$ for every operator b commuting with $a_1, \ldots, a_n$, and any $x \in X$.
4. $sp(a, y) \subseteq sp(a, Y)$ whenever Y is a (linear closed) subspace of X, invariant for $a_1, \ldots, a_n$, and $y \in Y$.

In order to obtain a minimal restriction of the integrand in (0.3) it is necessary to have a global solution ψ for the equation $sx = (\alpha \oplus \bar{\partial})\psi$; but in the case $n = 1$, such a solution does not necessarily exist without some restrictive assumptions [4, Remark 1.2].

We shall prove that a global solution exists if the following condition is fulfilled.

(g) $H^{n-1}(\mathscr{B}(G, X), \alpha \oplus \bar{\partial}) = 0$ for any open set $G \subset \mathbb{C}^n$.

Remark 1.2. If $n = 1$, this condition signifies that the operator a has the single-valued extension property [5; 3, Chap. 1, Definition 1.1].

To prove the global existence theorem (Theorem 1.1) we need the following lemmas.

LEMMA 1.1. *Let V_1, V_2 be two open sets in $\mathbb{C}^n$ such that $V_1 \cap V_2 \neq \varnothing$. Then for every $f \in \mathscr{B}(V_1 \cap V_2, X)$ there exists $f_i \in \mathscr{B}(V_i, X)$ $(i = 1, 2)$, satisfying $f = f_1 + f_2$ on $V_1 \cap V_2$.*

Proof. Consider $V = V_1 \cup V_2$ and let $\{W_j\}$ be a locally finite covering of V subordinate to $\{V_1, V_2\}$ and $\{h_j\}$ be a C^∞-partition of unity corresponding to $\{W_j\}$. For all indices j such that $W_j \subset V_1$, we define f_{j2} on V_2 by

$$f_{j2}(z) = \begin{cases} h_j(z)\, f(z), & z \in W_j \cap V_2, \\ 0, & z \in V_2 \backslash (W_j \cap V_2). \end{cases}$$

Since the support of h_j is compact and contained in W_j, f_{j2} locally belongs to $\mathscr{B}(V_2, X)$ and therefore also globally [10, Definition 2.4]. For the other indices j we shall have $W_j \subset V_2$, and thus one may define the function f_{j1} on V_1 by

$$f_{j1}(z) = \begin{cases} h_j(z) f(z), & z \in W_j \cap V_1, \\ 0, & z \in V_1 \backslash (W_j \cap V_1), \end{cases}$$

which belongs to $\mathscr{B}(V_1, X)$. Since the covering $\{W_j\}$ is locally finite, the functions $f_1 = \sum f_{j1}$ and $f_2 = \sum f_{j2}$ are well defined and belong to $\mathscr{B}(V_1, X)$ respectively to $\mathscr{B}(V_2, X)$; finally, for $z \in V_1 \cap V_2$ we have $f(z) = \sum_j h_j(z) f(z) = f_1(z) + f_2(z)$.

Remark 1.3. Of course, one may also write f in the form $f = f_1' - f_2'$.

In the following we shall assume that (g) is fulfilled.

LEMMA 1.2. *Let V_i $(i = 1, 2)$ be two open sets in $\mathbb{C}^n$ such that there exist $\psi_i \in \Lambda^{n-1}[\sigma \cup dz, \mathscr{B}(V_i, X)]$ $(i = 1, 2)$, satisfying $sx = (\alpha \oplus \bar{\partial})\, \psi_i$ on V_i. Then there exists $\psi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathscr{B}(V_1 \cup V_2, X)]$ such that $sx = (\alpha \oplus \bar{\partial})\psi$ on $V_1 \cup V_2$.*

Proof. If $V_1 \cap V_2 = \varnothing$, we may define

$$\psi(z) = \begin{cases} \psi_1(z), & z \in V_1, \\ \psi_2(z), & z \in V_2. \end{cases}$$

Otherwise, we have $(\alpha \oplus \bar{\partial})[\psi_1 - \psi_2] = 0$ on $V_1 \cap V_2$. Since (g) is fulfilled with $G = V_1 \cap V_2$, there exists a form $\varphi \in \Lambda^{n-2}[\sigma \cup d\bar{z}, \mathscr{B}(V_1 \cap V_2, X)]$ such that $\psi_1 - \psi_2 = (\alpha \oplus \bar{\partial})\varphi$; by applying Remark 1.3 to the coefficients of φ, it follows that one may write φ in the form $\varphi = \varphi_2 - \varphi_1$, where $\varphi_i \in \Lambda^{n-2}[\sigma \cup d\bar{z}, \mathscr{B}(V_i, X)]$ $(i = 1, 2)$. Then $\psi_1 - \psi_2 = (\alpha \oplus \bar{\partial})[\varphi_2 - \varphi_1]$ on $V_1 \cap V_2$, hence $\psi_1 + (\alpha \oplus \bar{\partial})\, \varphi_1 = \psi_2 + (\alpha \oplus \bar{\partial})\, \varphi_2$ on $V_1 \cap V_2$; we have only to denote $\psi_i' = \psi_i + (\alpha \oplus \bar{\partial})\, \varphi_i$ $(i = 1, 2)$ and we shall obtain $sx = (\alpha \oplus \bar{\partial})\, \psi_i'$ on V_i $(i = 1, 2)$ and, moreover, $\psi_1' = \psi_2'$ on $V_1 \cap V_2$; consequently, by defining

$$\psi(z) = \begin{cases} \psi_1'(z), & z \in V_1, \\ \psi_2'(z), & z \in V_2, \end{cases}$$

we shall have the wanted form.

COROLLARY 1.1. *Let $\{V_i\}_{i=1}^m$ be a finite family of open sets such that on each of them, the equation in ψ, $sx = (\alpha \oplus \bar{\partial})\psi$ has a solution. Then there exists a solution on their union.*

COROLLARY 1.2. *For any compact set* $K \subset r(a, x)$, *there exists an open neighborhood* V *such that the equation* $sx = (\alpha \oplus \bar{\partial})\psi$ *has a solution* ψ *on* V.

THEOREM 1.1. *For every* $x \in X$, *the equation* $sx = (\alpha \oplus \bar{\partial})\psi$ *has a global solution* $\psi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathscr{B}(r(a, x), X)]$; *thus the resolvent set* $r(a, x)$ *is the maximal open set on which the equation* $sx = (\alpha \oplus \bar{\partial})\psi$ *has a solution.*

Proof. We shall consider an increasing sequence of compact sets K_ν, such that $r(a, x) = \bigcup_{\nu=1}^{\infty} K_\nu$. We intend to show that there exists a corresponding sequence of forms

$$\psi_\nu \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathscr{B}(r(a, x), X)]$$

such that $sx = (\alpha \oplus \bar{\partial})\,\psi_\nu$ in a neighborhood of K_ν and $\psi_{\nu+1} = \psi_\nu$ in a neighborhood of K_ν. Then $\lim_{\nu\to\infty} \psi_\nu = \psi$ is just a global solution.

We begin with K_1. By Corollary 1.2 there exists a neighborhood of this set and a form $\psi_1{}^*$ on this neighborhood such that $sx = (\alpha \oplus \bar{\partial})\,\psi_1{}^*$. Since $\mathscr{B}(r(a, x), X)$ is closed with respect to multiplication by C^∞-functions, [10, Theorem 2.16], we may find a form $\psi_1 \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathscr{B}(r(a, x), X)]$ equal to $\psi_1{}^*$ in a neighborhood of K_1; thus $sx = (\alpha \oplus \bar{\partial})\,\psi_1$ in a neighborhood of K_1.

Let us suppose that the forms $\psi_1, \ldots, \psi_i$ have been determined and let us define ψ_{i+1}. By Corollary 1.2, there exists a neighborhood of K_{i+1} and a form ψ_{i+1}^* on this neighborhood, such that $sx = (\alpha \oplus \bar{\partial})\,\psi_{i+1}^*$; moreover, we may suppose, as we have already seen, that $\psi_{i+1}^* \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathscr{B}(r(a, x), X)]$. Since $sx = (\alpha \oplus \bar{\partial})\,\psi_i$ in a neighborhood of K_i and $sx = (\alpha \oplus \bar{\partial})\,\psi_{i+1}^*$ in a neighborhood of K_{i+1}, we have $(\alpha \oplus \bar{\partial})[\psi_{i+1}^* - \psi_i] = 0$ in a neighborhood V of K_i; by (g) there exists a form φ' such that $\psi_{i+1}^* - \psi_i = (\alpha \oplus \bar{\partial})\,\varphi'$ on V. We may find a form φ equal to φ' in a neighborhood W of K_i, defined on $r(a, x)$. The form $\psi_{i+1} = \psi_{i+1}^* - (\alpha \oplus \bar{\partial})\varphi$ satisfies $sx = (\alpha \oplus \bar{\partial})\,\psi_{i+1}$ on the same open set as ψ_{i+1}^* and, moreover, $\psi_{i+1} = \psi_i$ on a neighborhood of K_i. This completes the proof.

COROLLARY 1.3. $sp(a, x) = \varnothing$ *implies* $x = 0$.

Proof. Indeed $sp(a, x) = \varnothing$ implies $r(a, x) = \mathbb{C}^n$; therefore, by Theorem 1.1, there exists a form $\psi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathscr{B}(\mathbb{C}^n, X)]$ such that $sx = (\alpha \oplus \bar{\partial})\psi$. On account of the definition of R_{z-a} (see the preliminaries) and by (0.3), we obtain $x = 0$.

The following Proposition 1.1 shows that our local spectrum is smaller than the so-called local analytic spectrum [1]. Before stating it let us recall the following definition [1].

DEFINITION 1.2. The analytic resolvent set of x with respect to a is the set of all $z \in \mathbb{C}^n$ such that there exist an open neighborhood V of z and n X-valued analytic functions $f_1, \ldots, f_n$ on V, satisfying: $x \equiv (\zeta_1 - a_1) f_1(\zeta) + \cdots + (\zeta_n - a_n) f_n(\zeta)$, $\zeta \in V$. The complement, $\mathbb{C}^n \backslash \rho(a, x)$, is called the local analytic spectrum of x with respect to a and is denoted by $\sigma(a, x)$.

PROPOSITION 1.1. *For every $x \in X$, we have $sp(a, x) \subseteq \sigma(a, x)$.*

Proof. Of course, we show the inclusion $\rho(a, x) \subseteq r(a, x)$. Let $z \in \rho(a, x)$ and let V be an open neighborhood of z as in Definition 1.2. If $f_1, \ldots, f_n$ are the functions occurring in this definition, we shall consider the form $\psi(\zeta) = \sum_{i=1}^{n} (-1)^{i-1} f_i(\zeta)\, s_1 \wedge \cdots \wedge \hat{s}_i \wedge \cdots \wedge s_n$ as an element of $\Lambda^{n-1}[\sigma \cup d\bar{z}, \mathscr{B}(V, X)]$ and it is sufficient to remark that $sx = (\alpha \oplus \bar{\partial})\psi$, to conclude $z \in r(a, x)$.

COROLLARY 1.4. $\sigma(a, x) = \varnothing$ *implies* $x = 0$.

We end this section with a local variant of formula (0.3) which will be necessary in Section 3. Let U be an open neighborhood of $sp(a, x)$; we shall prove that there exists a form $\chi \in \Lambda^n[\sigma \cup d\bar{z}, \mathscr{B}_0(\mathbb{C}^n, X)]$ such that $\operatorname{supp} \chi \subset U$ and the forms sx and χ have the same cohomology classes with respect to $(\alpha \oplus \bar{\partial})$. Assuming this fact already proved and taking into account formula (0.3), we obtain:

$$x = \frac{1}{(2\pi i)^n} \int_U (-1)^n \pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n . \tag{0.3'}$$

Let us prove now the existence of χ. By Theorem 1.1, there exists a form $\psi \in \Lambda^{n-1}[\sigma \cup dz, \mathscr{B}(r(a, x), X)]$ such that $sx = (\alpha \oplus \bar{\partial})\psi$. Let V be a relatively compact neighborhood of $sp(a, x)$ such that $\bar{V} \subset U$ and let φ be a scalar C^∞-function, $\varphi = 1$ outside of V and $\varphi = 0$ in a neighborhood of $sp(a, x)$; we may define the form

$$\tilde{\psi} = \begin{cases} \varphi\psi & \text{on } U \cap r(a, x), \\ 0 & \text{otherwise.} \end{cases}$$

If we denote $\chi = sx - (\alpha \oplus \bar{\partial})\tilde{\psi}$, this form is zero in $\mathbb{C}^n \backslash \bar{V}$, since

$$sx - (\alpha \oplus \bar{\partial})\, \tilde{\psi} = sx - (\alpha \oplus \bar{\partial})\, \varphi\psi = sx - (\alpha \oplus \bar{\partial})\, \psi = 0;$$

consequently $\operatorname{supp} \chi$ is compact and $\operatorname{supp} \chi \subset U$.

Remark 1.4. The formula (0.3′) is valid for every U and any form χ with the specified properties.

2. Cohomological Results

The first theorem of this section deals with the cohomology analytical groups in a neighborhood of the Taylor spectrum.

THEOREM 2.1. *Let $D = D_1 X \cdots X D_n$ be an open polydisc containing $sp(a, X)$. Then $H^i(\mathscr{U}(D, X), \alpha) = 0$ for any i, $0 \leqslant i \leqslant n - 1$.*

Proof. Using the duality between the cochain complex $F(X, a)$ and the Koszul complex $E(X, a)$, [10, Sect. 1], it is enough to show that $H_p(E(\tilde{X}, \tilde{a})) = 0$ for every p, $0 < p \leqslant n$, where we have denoted $\tilde{X} = \mathscr{U}(D, X)$ and $(\tilde{a}_j f)(z) = (z_j - a_j) f(z)$, $z \in D$, $1 \leqslant j \leqslant n$. Then, applying Lemma 1.4 from [9], it is sufficient to prove that $\tilde{a}_p$ has null kernel on the factor space $\tilde{X}/\tilde{X}_{p-1}$, $1 \leqslant p \leqslant n$, where $\tilde{X}_p = \tilde{a}_1 \tilde{X} + \cdots + \tilde{a}_p \tilde{X}$ and $\tilde{X}_0 = (0)$.

We begin with $p = 1$. Let $f \in \tilde{X}$ such that $\tilde{a}_1 f = 0$; this means $(z_1 - a_1) f(z) \equiv 0$, $z \in D$. Since $sp(a, X) \subset D_1 X \cdots X D_n$, taking into account the projection property of the Taylor spectrum [9, Lemma 3.1], we obtain $\pi_1 sp(a, X) = sp(a_1, X) \subset D_1$ (π_1 denotes the first coordinate projection); consequently, $f(z) = 0$ for $z \in D$ such that $z_1 \in r(a_1, X)$, hence $f(z) \equiv 0$, $z \in D$ (by the uniqueness of the analytic continuation).

Now let $f_1, \ldots, f_{p+1} \in \tilde{X}$ such that $\tilde{a}_{p+1} f_{p+1} = \tilde{a}_1 f_1 + \cdots + \tilde{a}_p f_p$ and let us prove that there exist $g_1, \ldots, g_p$ such that $f_{p+1} = \tilde{a}_1 g_1 + \cdots + \tilde{a}_p g_p$. Since $sp(a, X) \subset D_1 X \cdots X D_n$, again by the projection property of the spectrum, we obtain $sp((a_1, \ldots, a_p), X) \subset D_1 X \cdots X D_p$. Now, the equality $\tilde{a}_{p+1} f_{p+1} = \tilde{a}_1 f_1 + \cdots + \tilde{a}_p f_p$ means explicitly

$$(z_{p+1} - a_{p+1}) f_{p+1}(z) = (z_1 - a_1) f_1(z) + \cdots + (z_p - a_p) f_p(z), \qquad z \in D,$$

therefore by [10, Proposition 3.7], the Cauchy–Weil integral of X-valued analytic function $(z_{p+1} - a_{p+1}) f_{p+1}(z)$ (in the variables $z_1, \ldots, z_p$) with respect to p-tuple $(a_1, \ldots, a_p)$ is equal to zero. If we choose the circumferences $\Gamma_j \subset D_j$ ($j = 1, \ldots, p$), then by [10, Corollary 3.15] it follows that

$$\int_{\Gamma_1} \cdots \int_{\Gamma_p} (z_1 - a_1)^{-1} \cdots (z_p - a_p)^{-1} (z_{p+1} - a_{p+1}) f_{p+1}(z)\, dz_1 \cdots dz_p = 0$$

for every $z_i \in D_i$, $p + 1 \leqslant i \leqslant n$. From here, because a is a commuting n-tuple, we deduce:

$$(z_{p+1} - a_{p+1}) \int_{\Gamma_1} \cdots \int_{\Gamma_p} (z_1 - a_1)^{-1} \cdots (z_p - a_p)^{-1} f_{p+1}(z)\, dz_1 \cdots dz_p = 0,$$

hence

$$\int_{\Gamma_1} \cdots \int_{\Gamma_p} (z_1 - a_1)^{-1} \cdots (z_p - a_p)^{-1} f_{p+1}(z)\, dz_1 \cdots dz_p = 0$$

for any $z_i \in D_i$, $p + 1 \leqslant i \leqslant n$. Let us now write the Cauchy integral formula for the function $f_{p+1}(z)$ in the variables $z_1, \ldots, z_p$. We obtain

$$\begin{aligned} &f_{p+1}(z_1, \ldots, z_p, z_{p+1}, \ldots, z_n) \\ &\quad = \frac{1}{(2\pi i)^p} \int_{\Gamma_1} \cdots \int_{\Gamma_p} \frac{f_{p+1}(\zeta_1, \ldots, \zeta_p, z_{p+1}, \ldots, z_n)}{(\zeta_1 - z_1) \cdots (\zeta_p - z_p)}\, d\zeta_1 \cdots d\zeta_p, \end{aligned}$$

where z_i is inside of Γ_i $(1 \leqslant i \leqslant p)$. We shall put the integrand in a suitable form using the identity

$$\begin{aligned} &f_{p+1}(\zeta_1, \ldots, \zeta_p, z_{p+1}, \ldots, z_n) \\ &\quad = (\zeta_1 - a_1) \cdots (\zeta_p - a_p)(\zeta_1 - a_1)^{-1} \cdots (\zeta_p - a_p)^{-1} \\ &\qquad \times f_{p+1}(\zeta_1, \ldots, \zeta_p, z_{p+1}, \ldots, z_n) \\ &\quad = [(\zeta_1 - z_1) + (z_1 - a_1)] \cdots [(\zeta_p - z_p) + (z_p - a_p)] \\ &\qquad \times (\zeta_1 - a_1)^{-1} \cdots (\zeta_p - a_p)^{-1} \cdot f_{p+1}(\zeta_1, \ldots, \zeta_p, z_{p+1}, \ldots, z_n). \end{aligned}$$

By performing the product of the first factors we obtain terms containing at least a factor $z_i - a_i$, except the term $(\zeta_1 - z_1) \cdots (\zeta_p - z_p)$; the integral corresponding to it is

$$\frac{1}{(2\pi i)^p} \int_{\Gamma_1} \cdots \int_{\Gamma_p} \frac{\begin{bmatrix}(\zeta_1 - z_1) \cdots (\zeta_p - z_p)(\zeta_1 - a_1)^{-1} \cdots \\ (\zeta_p - a_p)^{-1} f_{p+1}(\zeta_1, \ldots, \zeta_p, z_{p+1}, \ldots, z_n)\end{bmatrix}}{(\zeta_1 - z_1) \cdots (\zeta_p - z_p)}\, d\zeta_1 \cdots d\zeta_p = 0.$$

All the factors $z_i - a_i$ of the other terms may be put outside of the corresponding integrals; the integrals which remain are analytic functions when z_i is inside of Γ_i $(1 \leqslant i \leqslant p)$ and $z_j \in D_j$ $(p + 1 \leqslant j \leqslant n)$. Since these integrals are not modified by the dilatation of the circumferences Γ_i, we have in fact an equality of the form $f_{p+1}(z) = (z_1 - a_1)\, g_1(z) + \cdots + (z_p - a_p)\, g_p(z)$, $z \in D$ and the proof is completed.

Remark 2.1. The conclusion remains true if $D = D_1 \times \cdots \times D_n$ is a polydomain instead of a polydisc.

Remark 2.2. If $X \neq (0)$ then $H^n(\mathscr{U}(D, X), \alpha) \neq 0$ for every open polydisc (or polydomain) D containing $sp(a, X)$. Indeed, otherwise it would follow that for every $x \in X$, there exist X-valued analytic functions $f_1, \ldots, f_n$ on D satisfying:

$$x = (z_1 - a_1) f_1(z) + \cdots + (z_p - a_p) f_p(z), \qquad z \in D,$$

and then by (0.3) and [10, Proposition 3.7] it follows that $x = 0$, thus contradicting the hypothesis $X \neq (0)$.

Remark 2.3. If $X \neq (0)$ and $n > 1$ then there exist open sets $V \subset r(a, X)$ such that $H^n(\mathscr{U}(V, X), \alpha) \neq 0$. Indeed it is enough to consider an open polydisc D containing $sp(a, X)$ and to denote by V the set $D \cap r(a, X)$. Then our statement is obtained by applying Hartogs' continuation theorem and the preceding remark. This is in contrast with the situation in the case $n = 1$; in this case, for any open set $V \subset r(a, X)$, we have $H^1(\mathscr{U}(V, X), \alpha) = 0$. However the following proposition is general.

PROPOSITION 2.1. *For any open set $V \subset r(a, X)$ we have*:

$$H^0(\mathscr{U}(V, X), \alpha) = 0, \qquad H^1(\mathscr{U}(V, X), \alpha) = 0.$$

Proof. This results from the left exactness of the global section functor, by taking into account that if V is a polydisc we already know that $H^0(\mathscr{U}(V, X), \alpha) = 0$, $H^1(\mathscr{U}(V, X), \alpha) = 0$ [9, Theorem 2.2 and Definition 1.1]. We shall also give a direct proof. First, if $f \in \mathscr{U}(V, X)$ and $\alpha f = 0$, then $f(z) = 0$ on every open polydisc contained in V, hence $f = 0$. Now let $\psi \in \Lambda^1[\sigma, \mathscr{U}(V, X)]$ such that $\alpha\psi = 0$. Then, for every open polydisc D contained in V we can find $f \in \mathscr{U}(D, X)$ such that $\psi = \alpha f$; we consider an open covering of V by polydiscs $V = \bigcup_{i \in I} D_i$; let $f_i \in \mathscr{U}(D_i, X)$ such that $\psi = \alpha f_i$ on D_i ; by substraction we obtain $\alpha(f_i - f_j) = 0$ on $D_i \cap D_j$, hence $f_i = f_j$ on $D_i \cap D_j$. This allows us to define an analytic function $f \in \mathscr{U}(V, X)$ by $f = f_i$ on D_i ; obviously, $\psi = \alpha f$.

The following theorem shows (in particular) how one may verify that $H^{n-1}(\mathscr{B}(G, X), \alpha \oplus \bar{\partial}) = 0$. Before stating it, let us give a variant of the Dolbeault–Grothendieck lemma, which will be necessary in the proof.

LEMMA 2.1. *Let D be an open polydisc and q be an integer, $0 \leqslant q \leqslant n - 1$. For every $\psi \in \Lambda^{q+1}[d\bar{z}, \mathscr{B}(D, X)]$, $\bar{\partial}_{\psi=0}$, there exists $\varphi \in \Lambda^q[d\bar{z}, \mathscr{B}(D, X)]$ such that $\psi = \bar{\partial}\varphi$.*

Proof. In the same way as in [8, Theorem 2.3.3] one may show that for every open relatively compact polydisc $D' \subset D$, there exists a form $\varphi \in \Lambda^q[d\bar{z}, \mathscr{B}(D, X)]$ such that $\psi = \bar{\partial}\varphi$ on D' (see also [10, Lemma 2.9]). Now the proof can be continued as in [8, Theorem 2.7.8].

THEOREM 2.2. *If* $H^i(\mathscr{U}(D, X), \alpha) = 0,\ 0 \leqslant i \leqslant n-1$, *for any open polydisc* D, *then* $H^i(\mathscr{B}(G, X), \alpha \oplus \bar{\partial}) = 0,\ 0 \leqslant i \leqslant n-1$ *for every open set* G.

Among the partial results obtained in the proof, we shall retain, as interesting by itself, the following.

PROPOSITION 2.1. *If* $H^i(\mathscr{U}(D, X), \alpha) = 0,\ \ 0 \leqslant i \leqslant n-1$, *for an open polydisc* D, *then* $H^i(\mathscr{B}(D, X), \alpha \oplus \bar{\partial}) = 0, 0 \leqslant i \leqslant n-1$.

Proof. Let $\psi \in \Lambda^p[\sigma \cup d\bar{z}, \mathscr{B}(D, X)]$ $(0 \leqslant p \leqslant n-1)$ such that $(\alpha \oplus \bar{\partial})\psi = 0$. If $p = 0$, then ψ is a function with the properties $\alpha\psi = 0$ and $\bar{\partial}\psi = 0$; the second property shows that ψ is an analytic function and, since $\alpha\psi = 0$ and $H^0(\mathscr{U}(D, X), \alpha) = 0$, it follows that $\psi = 0$. If $p > 0$ we can write $\psi = \psi_{0,p} + \psi_{1,p-1} + \cdots + \psi_{p-1,1} + \psi_{p,0}$ where the first index is the degree with respect to s and the second, the degree with respect to $d\bar{z}$. Since $(\alpha \oplus \bar{\partial})\psi = 0$, we obtain $\bar{\partial}\psi_{0,p} = 0,\ \alpha\psi_{0,p} + \bar{\partial}\psi_{1,p-1} = 0, \ldots, \alpha\psi_{p-1,1} + \bar{\partial}\psi_{p,0} = 0,\ \alpha\psi_{p,0} = 0$; let us start with $\psi_{0,p}$; this is a form of degree p in $d\bar{z}$, having coefficients in $\mathscr{B}(D, X)$; since $\bar{\partial}\psi_{0,p} = 0$ and D is a polydisc, by Lemma 2.1, there exists $\varphi_{0,p-1}$ such that $\psi_{0,p} = \bar{\partial}\varphi_{0,p-1}$; by replacing it in $\alpha\psi_{0,p} + \bar{\partial}\psi_{1,p-1} = 0$, we obtain $\alpha\bar{\partial}\varphi_{0,p-1} + \bar{\partial}\psi_{1,p-1} = 0$; hence $\bar{\partial}[\psi_{1,p-1} - \alpha\varphi_{0,p-1}] = 0$. Now consider the form $\psi_{1,p-1} - \alpha\varphi_{0,p-1}$ as a form in s having, as coefficients, forms of degree $p-1$ in $d\bar{z}$. Since $\bar{\partial}[\psi_{1,p-1} - \alpha\varphi_{0,p-1}] = 0$, it follows that the same equality will be satisfied by its coefficients, hence there exists a form $\varphi_{1,p-2}$ such that $\psi_{1,p-1} - \alpha\varphi_{0,p-1} = \bar{\partial}\varphi_{1,p-2}$, whence $\psi_{1,p-1} = \alpha\varphi_{0,p-1} + \bar{\partial}\varphi_{1,p-2}$. Arguing exactly as before, we shall obtain in the last but one step $\psi_{p-1,1} = \alpha\varphi_{p-2,1} + \bar{\partial}\varphi^*_{p-1,0}$, and from here, by replacing it in $\alpha\psi_{p-1,1} + \bar{\partial}\psi_{p,0} = 0$, we shall have: $\bar{\partial}[\psi_{p,0} - \alpha\varphi^*_{p-1,0}] = 0$. The form $\psi_{p,0} - \alpha\varphi^*_{p-1,0}$, being only in s, has analytic coefficients; since its degree is $p \leqslant n-1$, by assumption, there exists another form φ with analytic coefficients, such that $\psi_{p,0} - \alpha\varphi^*_{p-1,0} = \alpha\varphi$ (of course, it is necessary to observe that $\alpha[\psi_{p,0} - \alpha\varphi^*_{p-1,0}] = \alpha\psi_{p,0} = 0$). Therefore, in the last step $\psi_{p,0} = \alpha[\varphi^*_{p-1,0} + \varphi] = \alpha\varphi_{p-1,0}$ where we have denoted $\varphi_{p-1,0} = \varphi^*_{p-1,0} + \varphi$. Now it is easy to verify that

$$\psi = (\alpha \oplus \bar{\partial})[\varphi_{0,p-1} + \varphi_{1,p-2} + \cdots + \varphi_{p-2,1} + \varphi_{p-1,0}];$$

indeed $\psi_{0,p} = \bar{\partial}\varphi_{0,p-1}$, $\psi_{1,p-1} = \alpha\varphi_{0,p-1} + \bar{\partial}\varphi_{1,p-2}$, and so on, $\psi_{p-1,1} = \alpha\varphi_{p-2,1} + \bar{\partial}\varphi_{p-1,0}$ (φ has analytic coefficients), $\psi_{p,0} = \alpha\varphi_{p-1,0}$.

Remark 2.4. In fact, we have proved that if $H^i(\mathscr{U}(D, X), \alpha) = 0$ for an open polydisc D and a certain i, then $H^i(\mathscr{B}(D, X), \alpha \oplus \bar{\partial}) = 0$ with the same i.

The proof of the theorem. We shall prove the theorem step by step. Let us begin with $H^0(\mathscr{B}(G, X), \alpha \oplus \bar{\partial}) = 0$; if $f \in \mathscr{B}(G, X)$ and $(\alpha \oplus \bar{\partial}) f = 0$ then $\bar{\partial} f = 0$ (which signifies f is an analytic function) and $\alpha f = 0$; by assumption, f is then zero on every polydisc contained in G, therefore $f = 0$. Suppose that $H^{p-1}(\mathscr{B}(G, X), \alpha \oplus \bar{\partial}) = 0$ for any open set G $(1 < p \leqslant n - 1)$ and let us prove that $H^p(\mathscr{B}(G, X), \alpha \oplus \bar{\partial}) = 0$. Let $\psi \in \Lambda^p[\sigma \cup d\bar{z}, \mathscr{B}(G, X)$, such that $(\alpha \oplus \bar{\partial})\psi = 0$. Consider an increasing sequence $\{K_\nu\}_{\nu=1}^\infty$ of compact sets, such that $G = \bigcup_{\nu=1}^\infty K_\nu$. We shall define a sequence

$$\{\varphi_\nu\}_{\nu=1}^\infty, \ \varphi_\nu \in \Lambda^p[\sigma \cup d\bar{z}, \mathscr{B}(G, X)]$$

such that $\psi = (\alpha \oplus \bar{\partial})\, \varphi_\nu$ in a neighborhood of K_ν and $\varphi_\nu = \varphi_{\nu+1}$ in a neighborhood of K_ν. So, it will follow that

$$\varphi = \lim_{\nu \to \infty} \varphi_\nu \in \Lambda^p[\sigma \cup d\bar{z}, \mathscr{B}(G, X)] \qquad \text{and} \qquad \varphi = (\alpha \oplus \bar{\partial})\, \psi.$$

In order to define such a sequence, let us first remark that, by applying the preceding Proposition 2.1 and by arguing as in Lemma 1.2 (see also Corollary 1.2), we deduce that for every compact set $K \subset G$, there exists a neighborhood V and a form $\varphi \in \Lambda^{p-1}[\sigma \cup d\bar{z}, \mathscr{B}(V, X)]$ such that $\psi = (\alpha \oplus \bar{\partial})\varphi$; moreover, we can assume $\varphi \in \Lambda^{p-1}[\sigma \cup d\bar{z}, \mathscr{B}(G, X)]$. From here the argument follows as in the proof of Theorem 1.1.

Remark 2.5. A similar argument shows that in the definition of the functional analytic calculus [10, Sect. 4, Definition 4.2], the more elaborate space $\mathscr{B}$ may be replaced by the usual C^∞-space.

The last result of this section will be used in Section 4.

THEOREM 2.3. *If for a certain p, $0 \leqslant p \leqslant n - 1$, we have $H^{p+1}(X, \alpha(z)) = 0$ and $H^p(\mathscr{O}_z(X), \alpha) = 0$, where $\mathscr{O}_z(X)$ denotes the germs of X-valued analytic functions in a neighborhood of z, then $H^p(X, \alpha) = 0$.*

Proof. Let $\psi \in \Lambda^p[\sigma, X]$ such that $\alpha(z)\psi = 0$. We shall prove that there exists a form $\psi(\zeta)$ of degree p, analytic in a neighborhood V of z, such that $\alpha(\zeta)\, \psi(\zeta) \equiv 0$, $\zeta \in V$, and $\psi(z) = \psi$. Hence, from assumption $(H^p(\mathscr{O}_z(X), \alpha) = 0)$, it will follow that there exists another analytic form $\varphi(\zeta)$ (on a possibly smaller neighborhood of z) such that $\psi(\zeta) \equiv \alpha(\zeta)\, \varphi(\zeta)$. In particular for $\zeta = z$ we shall obtain $\psi = \psi(z) =$

$\alpha(z)\,\varphi(z)$ and the proof will be finished. Therefore it remains to prove the existence of the form $\psi(\zeta)$. We shall seek it, of course, as a Taylor series: $\psi(\zeta) = \psi + \sum_k' (\zeta - z)^k \psi_k$, where the coefficients ψ_k will be determined such that $\alpha(\zeta)\,\psi(\zeta) \equiv 0$ and to assure the convergence of the series. Since $H^{p+1}(X, \alpha(z)) = 0$, applying the inverse bounded theorem results in a constant $M > 0$, such that for any $\chi \in \Lambda^{p+1}[\sigma, X]$, $\alpha(z)\chi = 0$, the equation $\alpha(z)\theta = \chi$ has a solution $\theta \in \Lambda^p[\sigma, X]$ satisfying $\| \theta \| \leqslant M \| \chi \|$. Since

$$\alpha(\zeta) = \sum_{j=1}^{n} (\zeta_j - z_j)\, s_j + \alpha(z),$$

we have

$$\alpha(\zeta)\,\psi(\zeta) = \alpha(z)\,\psi + \sum_k{}' (\zeta - z)^k \alpha(z)\,\psi_k$$
$$+ \sum_{j=1}^{n} (\zeta_j - z_j)\, s_j \wedge \psi + \sum_k{}' \sum_{j=1}^{n} (\zeta - z)^{k+1_j} s_j \wedge \psi_k \equiv 0,$$

whence

$$\sum_k{}' (\zeta - z)^k \alpha(z)\,\psi_k + \sum_k{}' (\zeta - z)^k \sum_{j=1,\, k_j \neq 0}^{n} s_j \wedge \psi_{k-1_j} \equiv 0;$$

therefore $\alpha(z)\,\psi_k = -\sum_{j=1, k_j \neq 0} s_j \wedge \psi_{k-1_j}$ (here, of course, $\psi_0 = \psi$ and $1_j = (0,\ldots, 1_{(j)}, \ldots, 0)$). Whence the coefficients ψ_k can be determined step by step. Moreover, we want to have the inequalities $\| \psi_k \| \leqslant (Mn)^{|k|} \| \psi \|$ (the norm of a form

$$\theta = \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} x_{j_1 \cdots j_p}\, s_{j_1} \wedge \cdots \wedge s_{j_p}$$

is

$$\| \theta \| = \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} \| x_{j_1 \cdots j_p} \|).$$

For $k = 0$, $\psi_0 = \psi$ and the inequality is satisfied. Let us suppose the forms ψ_h with $h < k$ already determined and let us define ψ_k; we remark that

$$\alpha(z) \left[- \sum_{j=1,\, k_j \neq 0}^{n} s_j \wedge \psi_{k-1_j} \right] = \sum_{j=1,\, k_j \neq 0}^{n} s_j \wedge \alpha(z)\,\psi_{k-1_j}$$
$$= \sum_{j=1,\, k_j \neq 0}^{n} s_j \wedge \sum_{l=1,\, (k-1_j)_l \neq 0}^{n} s_l \wedge \psi_{k-1_j-1_l}$$
$$= \sum_{j=1,\, k_j \neq 0}^{n} \sum_{l=1,\, k_l \neq 0}^{n} s_j \wedge s_l \wedge \psi_{k-1_j-1_l} = 0,$$

hence, taking into account that $H^{p+1}(X, \alpha(z)) = 0$, there exists $\psi_k \in \Lambda^p[\sigma, X]$ such that

$$\alpha(z)\,\psi_k = -\sum_{j=1,\,k_j\neq 0}^{n} s_j \wedge \psi_{k-1_j}$$

and

$$\|\psi_k\| \leqslant M\Big\| -\sum_{j=1,\,k_j\neq 0}^{n} s_j \wedge \psi_{k-1_j}\Big\| \leqslant M \sum_{j=1,\,k_j\neq 0}^{n} \|\psi_{k-1_j}\|$$

$$\leqslant M \sum_{j=1,\,k_j\neq 0}^{n} (Mn)^{|k|-1}\|\psi\| \leqslant M^{|k|}n^{|k|}\|\psi\| = (Mn)^{|k|}\|\psi\|.$$

The proof is complete.

COROLLARY 2.1. *Assume that for a certain* $z \in \mathbb{C}^n$, *we have* $H^n(X, \alpha(z)) = 0$ *and* $H^p(\mathcal{O}_z(X), \alpha) = 0$, $0 \leqslant p \leqslant n-1$. *Then* $z \in r(a, X)$.

COROLLARY 2.2. *Assume that for a certain* $z \in \mathbb{C}^n$, *we have* $H^p(\mathcal{O}_z(X), \alpha) = 0$, $0 \leqslant p \leqslant n$. *Then* $z \in r(a, X)$.

Theorem 2.3 was suggested to me by the following.

Conjecture. If $H^i(\mathcal{U}(V, X), \alpha) = 0$, $0 \leqslant i \leqslant n$, for a neighborhood V of z, then $z \in r(a, X)$.

3. DECOMPOSABLE n-TUPLES

Let Ω be a topological space and X be a complex Banach space. We denote by $\mathscr{F}(\Omega)$ the family of all closed subsets of Ω, and by $\mathscr{S}(X)$, the family of all (linear closed) subspaces of X.

DEFINITION 3.1. (see [7]). A spectral capacity of (Ω, X) type is an application $\mathscr{E}: \mathscr{F}(\Omega) \to \mathscr{S}(X)$ satisfying the following conditions.

(i) $\mathscr{E}(\varnothing) = \{0\}$, $\mathscr{E}(\Omega) = X$.

(ii) $\mathscr{E}(\bigcap_{i\in I} F_i) = \bigcap_{i\in I} \mathscr{E}(F_i)$ for any family $\{F_i\}_{i\in I} \subset \mathscr{F}(\Omega)$.

(iii) If $\{G_j\}_{j=1}^m$ is an open covering of Ω, then $X = \sum_{j=1}^m \mathscr{E}(\overline{G}_j)$ (i.e. every $x \in X$ can be written, not necessarily in a unique manner, in the form $x = \sum_{j=1}^m y_j$, where $y_j \in \mathscr{E}(\overline{G}_j)$, $1 \leqslant j \leqslant m$).

DEFINITION 3.2. A commutative *n-tuple* $a = (a_1, ..., a_n)$ of linear continuous operators on X is decomposable if there exists a spectral capacity $\mathscr{E}$ of $(\mathbb{C}^n, X)$ type, such that

(iv) $a_i\mathscr{E}(F) \subseteq \mathscr{E}(F)$ for any $F \in \mathscr{F}(\mathbb{C}^n)$, $1 \leqslant i \leqslant n$.

(v) $sp(a, \mathscr{E}(F)) \subseteq F$ for any $F \in \mathscr{F}(\mathbb{C}^n)$.

PROPOSITION 3.1. *Let a be a decomposable n-tuple, D be an open polydisc, and p an integer, $0 \leqslant p \leqslant n - 1$. Also let $\psi \in \Lambda^p[\sigma, \mathscr{U}(D, X)]$ be a form such that $\alpha\psi = 0$. Then for any open polydisc D', $\bar{D}' \subset D$, there exists a form $\varphi \in \Lambda^{p-1}[\sigma, \mathscr{U}(D', X)]$ such that $\psi = \alpha\varphi$.*

Proof. Choose two closed sets F_1 and F_2 such that $F_1 \subset D$, F_2 is disjoint from $\bar{D}'$, and $X = \mathscr{E}(F_1) + \mathscr{E}(F_2)$. Thus we have the short exact sequence $0 \to \mathscr{E}(F_1) \cap \mathscr{E}(F_2) \to \mathscr{E}(F_1) \oplus \mathscr{E}(F_2) \to X \to 0$, where the first application is given by $x \to x \oplus (-x)$ and the second, by $x_1 \oplus x_2 \to x_1 + x_2$. Since D is a polydisc, the sequence

$$0 \to \mathscr{U}(D, \mathscr{E}(F_1) \cap \mathscr{E}(F_2)) \to \mathscr{U}(D, \mathscr{E}(F_1) \oplus \mathscr{E}(F_2)) \to \mathscr{U}(D, X) \to 0$$

is also exact (see [9, Theorem 2.2]). In particular, an X-valued analytic function f on D can be represented in the form $f = f_1 + f_2$ where f_i is an $\mathscr{E}(F_i)$-valued analytic function on D, $i = 1, 2$. Therefore we can write $\psi = \psi_1 + \psi_2$, where $\psi_i \in \Lambda^p[\sigma, \mathscr{U}(D, \mathscr{E}(F_i))]$. Thus $\alpha\psi = \alpha\psi_1 + \alpha\psi_2 = 0$, whence $\alpha\psi_1 = \alpha[-\psi_2]$. The form $\chi = \alpha\psi_1 = \alpha[-\psi_2]$ has the degree $p + 1$ and its coefficients are $\mathscr{E}(F_1) \cap \mathscr{E}(F_2)$-valued functions. We shall prove there exists a form

$$\psi^* \in \Lambda^p[\sigma, \mathscr{U}(D, \mathscr{E}(F_1 \cap F_2))]$$

such that $\chi = \alpha\psi^*$. Assuming this fact already proved, we shall obtain $\alpha\psi_1 = \alpha\psi^*$, hence $\alpha(\psi_1 - \psi^*) = 0$. Since $sp(a, \mathscr{E}(F_1)) \subseteq F_1 \subset D$ and the form $\psi_1 - \psi^*$ is of degree at most $n - 1$, having as coefficients $\mathscr{E}(F_1)$-valued analytic functions, by Theorem 2.1, there exists $\varphi_1 \in \Lambda^{p-1}[\sigma, \mathscr{U}(D, \mathscr{E}(F_1))]$ satisfying $\psi_1 - \psi^* = \alpha\varphi_1$. On the other hand, from the equality $\alpha[-\psi_2] = \alpha\psi^*$, we deduce $\alpha[\psi^* + \psi_2] = 0$; since $sp(a, \mathscr{E}(F_2)) \subseteq F_2$ and F_2 is disjoint from $\bar{D}'$, it follows $D' \subset r(a, \mathscr{E}(F_2))$ and therefore $H^i(\mathscr{U}(D', \mathscr{E}(F_2)), \alpha) = 0$ for every i, $0 \leqslant i \leqslant n$ [9, Theorem 2.2]. The form $\psi^* + \psi_2$ has as coefficients, $\mathscr{E}(F_2)$-valued analytic functions, hence there exists a form $\varphi_2 \in \Lambda^{p-1}[\sigma, \mathscr{U}(D', \mathscr{E}(F_2))]$ such that $\psi^* + \psi_2 = \alpha\varphi_2$ on D'. Denoting $\varphi_1 + \varphi_2$ by φ, we have $\psi = (\psi_1 - \psi^*) + (\psi^* + \psi_2) = \alpha\varphi$ on D'. Then it remains to prove the existence of the form ψ^*. If $p + 1 < n$ then the form χ has the degree at most $n - 1$ and its coefficients are analytic functions with values in $\mathscr{E}(F_1) \cap \mathscr{E}(F_2) = \mathscr{E}(F_1 \cap F_2)$; but $sp(a, \mathscr{E}(F_1 \cap F_2)) \subseteq F_1 \cap F_2 \subset D$ and D is an open polydisc, therefore by Theorem 2.1, there exists a form $\psi^* \in \Lambda^p[\sigma, \mathscr{U}(D, \mathscr{E}(F_1 \cap F_2))]$ such that $\chi = \alpha\psi^*$ (one takes into account that $\alpha\chi = 0$). The case $p + 1 = n$ is a little more complicated, but the proof is similar to that of Theorem 2.1. Let $\chi = fs_1 \wedge \cdots \wedge s_n$ where $f \in \mathscr{U}(D, \mathscr{E}(F_1 \cap F_2))$ and $D = D_1 X \cdots X D_n$. Since $\alpha\psi_1 = fs_1 \wedge \cdots \wedge s_n$ where

$$\psi_1 \in \Lambda^{n-1}[\sigma, \mathscr{U}(D, \mathscr{E}(F_1))] \quad \text{and} \quad sp(a, \mathscr{E}(F_1)) \subseteq F_1 \subset D,$$

the Cauchy–Weil integral of f on D with respect to $a/\mathscr{E}(F_1)$ is equal to zero. We shall consider the circumferences Γ_i in D_i containing inside the compact set $sp(a_i, \mathscr{E}(F_1)) \cup sp(a_i, \mathscr{E}(F_1 \cap F_2)) \subset D_i$, $1 \leqslant i \leqslant n$. Applying [10, Corollary 3.15] it follows that

$$\int_{\Gamma_1} \cdots \int_{\Gamma_n} \prod_{i=1}^{n} (\zeta_i - a_i/\mathscr{E}(F_1))^{-1} f(\zeta)\, d\zeta_1 \cdots d\zeta_n = 0.$$

In the same way as in the quoted theorem we deduce

$$f(z) = (z_1 - a_1)\, g_1(z) + \cdots + (z_n - a_n)\, g_n(z), \qquad \text{where} \quad g_1, \ldots, g_n$$

are analytic $\mathscr{E}(F_1 \cap F_2)$-valued functions on D, and the proof is completed.

THEOREM 3.1. *If a is decomposable then for any open polydisc D, we have $H^i(\mathscr{U}(D, X), \alpha) = 0$, $0 \leqslant i \leqslant n - 1$.*

Proof. We start with $i = 0$. Let $f \in \mathscr{U}(D, X)$ such that $\alpha f = 0$; then by the preceding Proposition 3.1 we have $f = 0$ on every open polydisc D', $\bar{D}' \subset D$, therefore $f = 0$. Let us now suppose that for any open polydisc D, $H^{p-1}(\mathscr{U}(D, X), \alpha) = 0$, $0 < p \leqslant n - 1$, and let us prove that $H^p(\mathscr{U}(D, X), \alpha) = 0$. Consider a sequence of open polydiscs D_ν, such that $\bar{D}_\nu \subset D_{\nu+1}$ and $D = \bigcup_\nu D_\nu$. Let $\psi \in \Lambda^p[\sigma, \mathscr{U}(D, X)]$, $\alpha\psi = 0$. Applying Proposition 3.1 for D_2, we can find $\varphi_1 \in \Lambda^{p-1}[\sigma, \mathscr{U}(D_2, X)]$ such that $\psi = \alpha\varphi_1$; for D_3 we can analogously find a form φ_2' such that $\psi = \alpha\varphi_2'$. Therefore $\alpha(\varphi_1 - \varphi_2') = 0$ on D_2, whence by applying the inductive assumption, there exists $\chi \in \Lambda^{p-1}[\sigma, \mathscr{U}(D_2, X)]$ such that $\varphi_1 - \varphi_2' = \alpha\chi$. Let us keep in Taylor expansion of χ on D_2 a sufficiently large number of terms such that, denoting this part by χ', we will have $\| \alpha\chi - \alpha\chi' \| < \frac{1}{2}$ on $\bar{D}_1$. Replacing φ_2' by $\varphi_2 = \varphi_2' + \alpha\chi'$ we obtain a form on D_3 such that $\alpha\varphi_2 = \psi$ and $\| \varphi_1 - \varphi_2 \| = \| \varphi_2' + \alpha\chi - \varphi_2' - \alpha\chi' \| < \frac{1}{2}$ on $\bar{D}_1$. Step by step we can define a sequence $\{\varphi_\nu\}_{\nu=1}^{\infty}$, $\varphi_\nu \in \Lambda^{p-1}[\sigma, \mathscr{U}(D_{\nu+1}, X)]$

with the properties: $\psi = \alpha\varphi_\nu$ on $D_{\nu+1}$ and $\| \varphi_{\nu+1} - \varphi_\nu \| < 1/2^\nu$ on $\bar{D}_\nu$. Obviously, the sequence $\{\varphi_\nu\}_{\nu=1}^\infty$ converges to a form φ having analytic coefficients on D, which satisfies $\psi = \alpha\varphi$, as desired.

Corollary 3.1. *If a is decomposable, then for any open set $G \subset \mathbb{C}^n$, we have $H^i(\mathscr{B}(G, X), \alpha \oplus \bar{\partial}) = 0$, $0 \leqslant i \leqslant n-1$.*

Proof. It is enough to combine Theorem 3.1 with Theorem 2.2.

Remark 3.1. If a is decomposable, then, in particular, we have $H^{n-1}(\mathscr{B}(G, X), \alpha \oplus \bar{\partial}) = 0$ for any open set G in $\mathbb{C}^n$; this allows us to define $sp(a, x)$ and to use all the results from the first section.

Theorem 3.2 (of uniqueness). *If a is a decomposable n-tuple then there exists a unique spectral capacity $\mathscr{E}$ of $(\mathbb{C}^n, X)$ type which satisfies the conditions* (iv) *and* (v) *of Definition* 3.2. *This capacity is given by $\mathscr{E}(F) = \{x, x \in X, sp(a, x) \subseteq F\}$ for any closed set F in $\mathbb{C}^n$.*

Proof. Let us denote $X_{[a]}(F) = \{x, x \in X, sp(a, x) \subseteq F\}$. The inclusion $\mathscr{E}(F) \subseteq X_{[a]}(F)$ is obvious, on account of Property (4) for $sp(a, x)$. In order to prove the inverse inclusion, it is enough to show that for every open set G, $F \subset G$, we have $X_{[a]}(F) \subseteq \mathscr{E}(\bar{G})$; thus it will follow that $X_{[a]}(F) \subseteq \bigcap_{F \subset G} \mathscr{E}(\bar{G}) = \mathscr{E}(\bigcap_{F \subset G} \bar{G}) = \mathscr{E}(F)$, as desired. Let F_1 and F_2 be two closed sets such that $X = \mathscr{E}(F_1) + \mathscr{E}(F_2)$ and $F \subset F_1 \subset G$, while $F_2 \cap F = \varnothing$. If $x \in X_{[a]}(F)$, let $x = x_1 + x_2$ be a decomposition with $x_i \in \mathscr{E}(F_i)$ $(i = 1, 2)$. By Theorem 1.1, there exists a form ψ on $r(a, x)$ and a form ψ_2 on $r(a, x_2)$ such that $sx \equiv (\alpha \oplus \bar{\partial})\psi$ and $sx_2 \equiv (\alpha \oplus \bar{\partial})\psi_2$; hence, on $r(a, x) \cap r(a, x_2)$ we have $sx_1 \equiv (\alpha \oplus \bar{\partial})[\psi - \psi_2]$. Now let V be an open neighborhood of F, disjoint from F_2 and φ be a C^∞-scalar function on V, equal to 1 outside of a relatively compact neighborhood of the set $sp(a, x)$ and to 0 in a neighborhood of this set; then we can define the product $\varphi\psi$ as a form on V, while ψ_2 is already defined on V $(x_2 \in \mathscr{E}(F_2))$, hence $sp(a, x_2) \subseteq F_2$, therefore $r(a, x_2) \supseteq \mathscr{C}F_2 \supseteq V$. The form $sx_1 - (\alpha \oplus \bar{\partial})[\varphi\psi - \psi_2]$ is defined on V and has compact support; indeed we can write $h_1 = sx_1 - (\alpha \oplus \bar{\partial})[\varphi\psi - \psi_2] = sx - (\alpha \oplus \bar{\partial})\varphi\psi - [sx_2 - (\alpha \oplus \bar{\partial})\psi_2] = sx - (\alpha \oplus \bar{\partial})\varphi\psi = h$ and the last form has compact support in V. Applying (0.3′) for x we obtain

$$x = \frac{1}{(2\pi i)^n} \int_V (-1)^n \pi h \wedge dz_1 \wedge \cdots \wedge dz_n$$

$$= \frac{1}{(2\pi i)^n} \int_V (-1)^n \pi h_1 \wedge dz_1 \wedge \cdots \wedge dz_n .$$

Now, we shall prove that one may choose h_1 such that its coefficients belong to $\mathscr{E}(F_1)$. Let us denote $\psi_1{}^* = \psi - \psi_2$ on $r(a, x) \cap r(a, x_2)$. Taking into account the exactness of the short sequence $0 \to \mathscr{E}(F_1 \cap F_2) \to \mathscr{E}(F_1) \oplus \mathscr{E}(F_2) \to X \to 0$, and applying [10, Theorem 2.16 (5)] we can write $\psi_1{}^* = \chi_1 + \chi_2$ where the forms χ_1 and χ_2 are defined on $r(a, x) \cap r(a, x_2)$ and their coefficients are $\mathscr{E}(F_1)$-valued, respectively $\mathscr{E}(F_2)$-valued functions. Thus $sx_1 = (\alpha \oplus \bar{\partial})\,\psi_1{}^* = (\alpha \oplus \bar{\partial})\,\chi_1 + (\alpha \oplus \bar{\partial})\,\chi_2$, whence $sx_1 - (\alpha \oplus \bar{\partial})\,\chi_1 = (\alpha \oplus \bar{\partial})\,\chi_2$. The form $sx_1 - (\alpha \oplus \bar{\partial})\,\chi_1$ is $\alpha \oplus \bar{\partial}$-closed on $r(a, x) \cap \mathscr{C}F_2$ and has coefficients in $\mathscr{E}(F_1 \cap F_2)$. Since $sp(a, \mathscr{E}(F_1 \cap F_2)) \subseteq F_1 \cap F_2$, the open set $r(a, x) \cap \mathscr{C}F_2$ is contained in $r(a, \mathscr{E}(F_1 \cap F_2))$, therefore the preceding form is also $\alpha \oplus \bar{\partial}$-exact; then there exists a form χ having the coefficients in $\mathscr{E}(F_1 \cap F_2)$, satisfying $sx_1 - (\alpha \oplus \bar{\partial})\,\chi_1 = (\alpha \oplus \bar{\partial})\chi$, whence $sx_1 = (\alpha \oplus \bar{\partial})(\chi_1 + \chi)$. The form $\chi_1 + \chi$ has the coefficients in $\mathscr{E}(F_1)$ and by multiplying it by a suitable C^∞-scalar function, we obtain an equality of the form $sx_1 - (\alpha \oplus \bar{\partial})\,\chi_1{}^* = h_1{}^*$, where $h_1{}^*$ has compact support and its coefficients belong to $\mathscr{E}(F_1)$. Consider together the equalities $sx_1 - (\alpha \oplus \bar{\partial})\,\psi_1{}^* = h_1$ and $sx_1 - (\alpha \oplus \bar{\partial})\,\chi_1{}^* = h_1{}^*$ and, again by (0.3′), it follows that

$$\begin{aligned} x &= \frac{1}{(2\pi i)^n} \int_V (-1)^n\, \pi h_1 \wedge dz_1 \wedge \cdots \wedge dz_n \\ &= \frac{1}{(2\pi i)^n} \int_V (-1)^n\, \pi h_1{}^* \wedge dz_1 \wedge \cdots \wedge dz_n \in \mathscr{E}(F_1) \subset \mathscr{E}(\bar{G}), \end{aligned}$$

completing the proof.

COROLLARY 3.2. *If a is decomposable, then for any closed set F, $X_{[a]}(F)$ is a closed linear subspace of X and $sp(a, X_{[a]}(F)) \subseteq F$; furthermore, $X_{[a]}(F)$ contains any linear closed subspace Y, invariant for a, such that $sp(a, Y) \subseteq F$ (i.e. $X_{[a]}(F)$ is a maximal spectral space for a, see* [3, Definition 3.1]).

Proof. The first part is an immediate consequence of Theorem 3.2, while for the second part it is enough to remark that if $sp(a, Y) \subseteq F$ then for every $y \in Y$, $sp(a, y) \subseteq sp(a, Y) \subseteq F$, hence $y \in X_{[a]}(F)$.

COROLLARY 3.3. *If a is decomposable then for every $x \in X$, we have $sp(a, x) = \sigma(a, x)$.*

Proof. The inclusion $sp(a, x) \subseteq \sigma(a, x)$ has been proved in the first section (Proposition 1.1). Conversely, if $z \in r(a, x)$, let D be an open polydisc, $D \subset r(a, x)$ having the center in z; since $x \in X_{[a]}(sp(a, x))$

and $sp(a, X_{[a]}(sp(a, x))) \subseteq sp(a, x)$, D is contained in $r(a, X_{[a]}(sp(a, x)))$, therefore by [9, Theorem 2.2] there exist the X-valued analytic functions $f_1, ..., f_n$ on D, such that $x \equiv (\zeta_1 - a_1) f_1(\zeta) + \cdots + (\zeta_n - a_n) f_n(\zeta)$, $\zeta \in D$.

Corollary 3.4. *For any set H we have $X_a(H) = X_{[a]}(H)$ where $X_a(H) = \{x, x \in X, \sigma(a, x) \subseteq H\}$.*

Corollary 3.5. *For any $x \in X$, we have* $\operatorname{supp}(\mathscr{E}, x) = sp(a, x)$, *where* $\operatorname{supp}(\mathscr{E}, x) = \bigcap \{F; F$ *closed*, $x \in \mathscr{E}(F)\}$.

Proof. On the one hand, $\mathscr{E}(\operatorname{supp}(\mathscr{E}, x)) = \mathscr{E}(\bigcap \{F;\ F \text{ closed}, x \in \mathscr{E}(F)\}) = \bigcap \{\mathscr{E}(F);\ F \text{ closed}, x \in \mathscr{E}(F)\}$, hence $x \in \mathscr{E}(\operatorname{supp}(\mathscr{E}, x))$ and thus, $sp(a, x) \subseteq \operatorname{supp}(\mathscr{E}, x)$. On the other hand, $x \in X_{[a]}(sp(a, x)) = \mathscr{E}(sp(a, x))$, whence $\operatorname{supp}(\mathscr{E}, x) \subseteq sp(a, x)$.

Proposition 3.2. *If a is decomposable then $\bigcup_{x \in X} sp(a, x) = sp(a, X)$.*

Proof. The inclusion $\bigcup_{x \in X} sp(a, x) \subseteq sp(a, X)$ is obvious. Conversely, if $z \in \bigcap_{x \in X} r(a, x)$, then $H^n(\mathscr{O}_z(X), \alpha) = 0$, and by Theorem 3.1, we also have $H^i(\mathscr{U}(D, X), \alpha) = 0$, $0 \leqslant i \leqslant n - 1$, for any open polydisc D; taking into account Corollary 2.2, we obtain $z \in r(a, X)$.

Remark 3.1. We have, in fact, proved that if $H^i(\mathscr{U}(D, X), \alpha) = 0$, $0 \leqslant i \leqslant n - 1$, for any open polydisc D, then $\bigcup_{x \in X} sp(a, x) = sp(a, X)$. This result may be compared to [2, Theorem 2.1].

Proposition 3.3. *If a is decomposable and F_1, F_2 are two disjoint closed sets, then $X_a(F_1 \cup F_2) = X_a(F_1) \oplus X_a(F_2)$.*

For the case $n = 1$, see [2, Proposition 2.3].

Proof. First, let us remark that for every closed set F we have $X_a(F) = X_a(F^*)$ where $F^* = sp(a, X_a(F))$. Taking $F = F_1 \cup F_2$, we obtain $X_a(F_1 \cup F_2) = X_a(F^*)$, $F^* \subseteq F_1 \cup F_2$; because $sp(a, X_a(F^*)) = F^*$, the closed sets $F_1 \cap F^*$ and $F_2 \cap F^*$ are separated parts of the spectrum of a on the space $X_a(F^*)$, consequently, by [10, Theorem 4.9], there exist two linear closed subspaces X_1 and X_2 in $X_a(F^*)$ such that $X_a(F^*) = X_1 \oplus X_2$ and $sp(a, X_1) \subseteq F_1 \cap F^*$, $sp(a, X_2) \subseteq F_2 \cap F^*$; but $X_a(F_1) \subseteq X_a(F^*)$, $X_a(F_2) \subseteq X_a(F^*)$, $X_a(F_1) \cap X_a(F_2) = X_a(F_1 \cap F_2) = X_a(\varnothing) = \{0\}$, and, by Corollary 3.2, $X_1 \subseteq X_a(F_1)$, $X_2 \subseteq X_a(F_2)$; thus $X_a(F_1 \cup F_2) = X_a(F_1) \oplus X_a(F_2)$.

DEFINITION 3.3. The support of the spectral capacity $\mathscr{E}$ is the set $\operatorname{supp} \mathscr{E} = \bigcap \{F; F \text{ closed}, \mathscr{E}(F) = X\}$. Obviously, $\operatorname{supp} \mathscr{E}$ is the smallest closed set F such that $\mathscr{E}(F) = X$.

PROPOSITION 3.4. *If a is decomposable and $\mathscr{E}$ denotes its spectral capacity, then* $\operatorname{supp} \mathscr{E} = sp(a, X)$.

Proof. The inclusion $sp(a, X) \subseteq \operatorname{supp} \mathscr{E}$ is simple; indeed, for any closed set F satisfying $\mathscr{E}(F) = X$, we have $sp(a, X) = sp(a, \mathscr{E}(F)) \subseteq F$; this inclusion is, in particular, valid for $F = \operatorname{supp} \mathscr{E}$. For proving the inverse inclusion let $z_0 \in r(a, X)$; we have to show that $z_0 \notin \operatorname{supp} \mathscr{E}$. Let V be a neighborhood of z_0 and let F be a closed set such that $z_0 \notin F$ and $X = \mathscr{E}(F) + \mathscr{E}(\bar{V})$. If we prove that $\mathscr{E}(\bar{V}) = \{0\}$, then it will follow that $\mathscr{E}(F) = X$; but $z_0 \notin F$, whence $z_0 \notin \operatorname{supp} \mathscr{E}$. In order to prove that $\mathscr{E}(\bar{V}) = \{0\}$, let $x \in \mathscr{E}(\bar{V})$; since $\bar{V} \subset r(a, X)$, it follows $sx = (\alpha \oplus \bar{\partial})\psi$ in a neighborhood (namely $\mathscr{C}\bar{V}$) of $sp(a, X)$, therefore applying (0.3) we obtain $x = 0$.

Remark 3.2. One may also say that $\operatorname{supp} \mathscr{E}$ is the smallest closed set outside of which $\mathscr{E}$ is zero.

COROLLARY 3.6. *For any closed set F we have $sp(a, \mathscr{E}(F)) \subseteq sp(a, X)$.*

Proof. Indeed $\mathscr{E}(F \cap sp(a, X)) = \mathscr{E}(F) \cap \mathscr{E}(sp(a, X)) = \mathscr{E}(F) \cap X = \mathscr{E}(F)$, therefore $sp(a, \mathscr{E}(F)) = sp(a, \mathscr{E}(F \cap sp(a, X)) \subseteq F \cap sp(a, X) \subseteq sp(a, X)$.

Remark 3.3. In all the preceding arguments we have used, in fact, instead of Property (iii) from Definition 3.1, the following less restrictive property:

(iii′) For any open covering $\{G_1, G_2\}$ of $\mathbb{C}^n$ by two open sets, we have $X = \mathscr{E}(\bar{G}_1) + \mathscr{E}(\bar{G}_2)$.

The final result of this section (Theorem 3.3) is about the behavior of decomposable n-tuples with respect to functional analytic calculus. Let U be a neighborhood of $sp(a, X)$ and let f be a $\mathbb{C}^m$-valued analytic function on U. Then by Taylor functional analytic calculus [10, Sect. 4], $f(a)$ is an m-tuple of linear continuous operators on X. Let us also remark that if Y is a (closed linear) subspace of X, invariant for a (i.e. for $a_1, \ldots, a_n$) and such that $sp(a, Y) \subseteq U$, then Y is also invariant for $f(a)$ and $f(a/Y) = f(a)/Y$ (see the Preliminaries).

THEOREM 3.3. *If a is decomposable then $f(a)$ is also decomposable. The spectral capacities $\mathscr{E}$ of a and $\mathscr{E}^*$ of $f(a)$ are connected by $\mathscr{E}^*(F) = \mathscr{E}(f^{-1}(F) \cap sp(a, X))$ for any $F \in \mathscr{F}(\mathbb{C}^m)$.*

Proof. We shall consider the application $\mathscr{E}^*$ defined as in the statement of the theorem, and we shall prove that it is a spectral capacity for $f(a)$. First let us observe that this definition is correct since for any $F \in \mathscr{F}(\mathbb{C}^m)$, $f^{-1}(F)$ is a closed set in U and since $sp(a, X)$ is a compact set in U, the intersection $f^{-1}(F) \cap sp(a, X)$ is a closed set in $\mathbb{C}^n$. It remains to verify the properties from Definitions 3.1 and 3.2.

(i) $\mathscr{E}^*(\varnothing) = \mathscr{E}(\varnothing) = \{0\}$, $\mathscr{E}^*(\mathbb{C}^m) = \mathscr{E}(U \cap sp(a, X)) = \mathscr{E}(sp(a, X)) = X$.

(ii) For any family $\{F_i\}_{i \in I} \subset \mathscr{F}(\mathbb{C}^m)$, we have

$$\mathscr{E}^*\left(\bigcap_{i\in I} F_i\right) = \mathscr{E}\left(f^{-1}\left(\bigcap_{i\in I} F_i\right) \cap sp(a, X)\right)$$

$$= \mathscr{E}\left(\bigcap_{i\in I} f^{-1}(F) \cap sp(a, X)\right) = \bigcap_{i\in I} \mathscr{E}^*(F_i).$$

(iii) If $\{G_j\}_{j=1}^p$ is an open finite covering for $\mathbb{C}^m$, then $\{f^{-1}(G_j)\}_{j=1}^p$ is an open finite covering for U. Since $\mathscr{E}$ satisfies (iii) and supp $\mathscr{E} = sp(a, X)$, the result is

$$\sum_{j=1}^{p} \mathscr{E}(\overline{f^{-1}(G_j)}) = \sum_{j=1}^{p} \mathscr{E}(\overline{f^{-1}(G_j)} \cap sp(a, X))$$

$$= X \subseteq \sum_{j=1}^{p} \mathscr{E}(f^{-1}(\bar{G}_j)) \cap sp(a, X) = \sum_{j=1}^{p} \mathscr{E}^*(\bar{G}_j) \subseteq X,$$

whence $X = \sum_{j=1}^{p} \mathscr{E}^*(\bar{G}_j)$.

Property (iv) is obvious. Finally, $sp(f(a), \mathscr{E}(f^{-1}(F) \cap sp(a, X)) = f(sp(a, \mathscr{E}(f^{-1}(F) \cap sp(a, X)) \subseteq F$ and thus Property (v) is also verified. The proof is concluded.

4. Quasinilpotent Equivalence and Commutators

If X is (as in the preceding sections) a complex Banach space, let $L(X)$ be the algebra of its linear continuous operators. For $a \in L(X)$, let us denote by $l(a)$ the (linear continuous) operator on $L(X)$ defined as the left multiplication by a, and by $r(a)$ the operator defined as the right multiplication by a. We remark that for any $a, b \in L(X)$, the operators $l(a)$ and $r(a)$ commute. Following [3, Chap. 2, Sect. 3] we shall denote $c(a, b) = l(a) - r(b)$.

Now let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ two commutative n-tuples in $L(X)$. We shall denote $c(a, b) = (c(a_1, b_1), \ldots, c(a_n, b_n))$. One may easy verify that this is a commutative n-tuple. If $k = (k_1, \ldots, k_n)$ is an n-tuple of nonnegative integers, we define $c^k(a, b) = c^{k_1}(a_1, b_1) \cdots c^{k_n}(a_n, b_n)$ and $(a\backslash b)^{[k]} = c^k(a, b)\, id$, where we have denoted by id the identity operator on X. In the case $n = 1$, these expressions has been defined in [3, Chap. 1, Sect. 2 and Chap. 2, Sect. 3]. The following two lemmas contain two important calculus properties for these expressions.

LEMMA 4.1. *For any* $u \in L(X)$ *and* j, $1 \leqslant j \leqslant n$, *we have* $c^{k+1_j}(a, b)u = a_j c^k(a, b)u - c^k(a, b)\, ub_j$ *where* $1_j = (0, \ldots, 1_{(j)}, \ldots, 0)$.

Proof. Indeed,

$$\begin{aligned} c^{k+1_j}(a, b)\, u &= c^{k_1}(a_1, b_1) \cdots c^{k_j+1}(a_j, b_j) \cdots c^{k_n}(a_n, b_n)\, u \\ &= c(a_j, b_j)\, c^{k_1}(a_1, b_1) \cdots c^{k_j}(a_j, b_j) \cdots c^{k_n}(a_n, b_n)\, u \\ &= c(a_j, b_j)\, c^k(a, b)\, u = a_j c^k(a, b)\, u - c^k(a, b)\, ub_j\,. \end{aligned}$$

If we put $u = id$, we obtain $(a\backslash b)^{[k+1]} = a_j(a\backslash b)^{[k]} - (a\backslash b)^{[k]} b_j$.

LEMMA 4.2. *For every three commutative n-tuples a, a', a'', and any k, we have* $(a\backslash a'')^{[k]} = \sum_{p\leqslant k} \binom{k}{p} (a\backslash a')^{[p]}(a'\backslash a'')^{[k-p]}$ ($p \leqslant k$ means $p_j \leqslant k_j$, $1 \leqslant j \leqslant n$, and $\binom{k}{p} = k!/p!(k-p)! = (k_1! \cdots k_n!)/[p_1! \cdots p_n!\,(k_1 - p_1)! \cdots (k_n - p_n)!]$).

Proof. We shall give an inductive argument. The equality is obviously satisfied for $k = (0, \ldots, 0)$. Assume that it is satisfied for k and let us prove it for $k + 1_j$ $(1 \leqslant j \leqslant n)$. We have

$$\begin{aligned} (a\backslash a'')^{[k+1_j]} &= a_j(a\backslash a'')^{[k]} - (a\backslash a'')^{[k]}\, a_j'' \\ &= \sum_{p\leqslant k} \binom{k}{p} a_j(a\backslash a')^{[p]}(a'\backslash a'')^{[k-p]} - \sum_{p\leqslant k} \binom{k}{p} (a\backslash a')^{[p]}(a'\backslash a'')^{[k-p]}\, a_j''\,. \end{aligned}$$

If we replace

$$a_j(a\backslash a')^{[p]} = (a\backslash a')^{[p+1_j]} + (a\backslash a')^{[p]}\, a_j$$

and

$$(a'\backslash a'')^{[k-p]}\, a_j'' = a_j'(a'\backslash a'')^{[k-p]} - (a'\backslash a'')^{[k-p+1_j]}$$

we obtain

$$
\begin{aligned}
(a\backslash a'')^{[k+1_j]} &= \sum_{p\leqslant k}\binom{k}{p}(a\backslash a')^{[p+1_j]}(a'\backslash a'')^{[k-p]} \\
&\quad + \sum_{p\leqslant k}\binom{k}{p}(a\backslash a')^{[p]}\,a_j'(a'\backslash a'')^{[k-p]} \\
&\quad - \sum_{p\leqslant k}\binom{k}{p}(a\backslash a')^{[p]}\,a_j'(a'\backslash a'')^{[k-p]} \\
&\quad + \sum_{p\leqslant k}\binom{k}{p}(a\backslash a')^{[p]}(a'\backslash a'')^{[k-p+1_j]} \\
&= \sum_{p\leqslant k}\binom{k}{p}(a\backslash a')^{[p+1_j]}(a'\backslash a'')^{[k-p]} \\
&\quad + \sum_{p\leqslant k}\binom{k}{p}(a\backslash a')^{[p]}(a'\backslash a'')^{[k-p+1_j]} \\
&= \sum_{1_j\leqslant p\leqslant k+1_j}\binom{k}{p-1_j}(a\backslash a')^{[p]}(a'\backslash a'')^{[k+1_j-p]} \\
&\quad + \sum_{p\leqslant k}\binom{k}{p}(a\backslash a')^{[p]}(a'\backslash a'')^{[k+1_j-p]} \\
&= \sum_{1_j\leqslant p\leqslant k+1_j,\,p_j=k_j+1}\binom{k}{p-1_j}(a\backslash a')^{[p]}(a'\backslash a'')^{[k+1_j-p]} \\
&\quad + \sum_{1_j\leqslant p\leqslant k}\left[\binom{k}{p-1_j}+\binom{k}{p}\right](a\backslash a')^{[p]}(a'\backslash a'')^{[k+1_j-p]} \\
&\quad + \sum_{p\leqslant k,\,p_j=0}\binom{k}{p}(a\backslash a')^{[p]}(a'\backslash a'')^{[k+1_j-p]}.
\end{aligned}
$$

Let us remark now that for $p_j = k_j + 1$ we have

$$\binom{k}{p-1_j} = \binom{k_1}{p_1}\cdots\binom{k_j}{k_j}\cdots\binom{k_n}{p_n}$$

and

$$\binom{k+1_j}{p} = \binom{k_1}{p_1}\cdots\binom{k_j+1}{k_j+1}\cdots\binom{k_n}{p_n},$$

therefore $\binom{k}{p-1_j} = \binom{k+1_j}{p}$. Similarly for $p_j = 0$ we have

$$\binom{k}{p} = \binom{k_1}{p_1}\cdots\binom{k_j}{0}\cdots\binom{k_n}{p_n} \quad \text{and}$$

$$\binom{k+1_j}{p} = \binom{k_1}{p_1}\cdots\binom{k_j+1}{0}\cdots\binom{k_n}{p_n},$$

hence $\binom{k}{p} = \binom{k+1_j}{p}$. Finally, it is easy to verify that

$$\binom{k}{p-1_j} + \binom{k}{p} = \binom{k+1_j}{p} \quad \text{for} \quad 1_j \leqslant p \leqslant k.$$

If we now substitute now these coefficients in the formula from above, we get

$$\begin{aligned}(a\backslash a'')^{[k+1_j]} &= \sum_{1_j \leqslant p \leqslant k+1_j,\, p_j = k_j+1} \binom{k+1_j}{p} (a\backslash a')^{[p]} (a'\backslash a'')^{[k+1_j-p]} \\ &\quad + \sum_{p \leqslant k,\, p_j=0} \binom{k+1_j}{p} (a\backslash a')^{[p]} (a'\backslash a'')^{[k+1_j-p]} \\ &\quad + \sum_{1_j \leqslant p \leqslant k} \binom{k+1_j}{p} (a\backslash a')^{[p]} (a'\backslash a'')^{[k+1_j-p]} \\ &= \sum_{p \leqslant k+1_j} \binom{k+1_j}{p} (a\backslash a')^{[p]} (a'\backslash a'')^{[k+1_j-p]}\end{aligned}$$

and the proof is completed.

Remark 4.1. If we put in the preceding lemma $a' = (0,..., 0)$ and $a'' = b$ we obtain $(a\backslash b)^{[k]} = \sum_{p \leqslant k} \binom{k}{p} (-1)^{|k-p|} a^p b^{k-p}$, where $a^p = a_1^{p_1} \cdots a_n^{p_n}$ and $b^{k-p} = b^{k_1-p_1} \cdots b^{k_n-p_n}$.

DEFINITION 4.1. We say that two n-tuples a and b are quasinilpotent equivalent and we denote $a \sim^q b$ if $\lim_{|k| \to \infty} \|(a\backslash b)^{[k]}\|^{1/|k|} = \lim_{|k| \to \infty} \|(b\backslash a)^{[k]}\|^{1/|k|} = 0$.

Remark 4.2. If $a \sim^q b$ then $a_i \sim^q b_i$ for every i, $1 \leqslant i \leqslant n$. Indeed, for every natural number k we have $(a_i\backslash b_i)^{[k]} = (a\backslash b)^{[k_i]}$ where $k_i = (0,..., k_i ,..., 0)$, hence

$$\lim_{k \to \infty} \|(a_i\backslash b_i)^{[k]}\|^{1/k} = \lim_{k \to \infty} \|(b_i\backslash a_i)^{[k]}\|^{1/k} = 0.$$

Remark 4.3. If the n-tuples a and b commute (i.e. $a_j b_k = b_k a_j$ for any j, k) then $a \sim^q b$ iff $a_i - b_i$ is quasinilpotent for every i, $1 \leqslant i \leqslant n$.

We have to prove that if $a_i - b_i$ is quasinilpotent for every i, then $a \sim^q b$. First let us remark that $(a\backslash b)^{[k]} = (a-b)^k = (a_1 - b_1)^{k_1} \cdots (a_n - b_n)^{k_n}$, hence $\|(a\backslash b)^{[k]}\| \leqslant \|(a_1 - b_1)^{k_1}\| \cdots \|(a_n - b_n)^{k_n}\|$. For any $\varepsilon > 0$, there exist some constants $M_1 = M_1(\varepsilon),..., M_n = M_n(\varepsilon)$ such that $\|(a_i - b_i)^{k_i}\| \leqslant M_i \varepsilon^{k_i}$, $1 \leqslant i \leqslant n$, whence $\|(a\backslash b)^{[k]}\| \leqslant M_1 \cdots M_n \varepsilon^{|k|}$, therefore $\|(a\backslash b)^{[k]}\|^{1/|k|} \leqslant (M_1 \cdots M_n)^{1/|k|} \varepsilon$; hence it follows that $\overline{\lim}_{|k|\to\infty} \|(a\backslash b)^{[k]}\|^{1/|k|} \leqslant \varepsilon$ and consequently, $\varepsilon > 0$ being arbitrary,

$$\lim_{|k|\to\infty} \|(a\backslash b)^{[k]}\|^{1/|k|} = \lim_{|k|\to\infty} \|(b\backslash a)^{[k]}\|^{1/|k|} = 0.$$

We shall now study the behavior of the Taylor spectrum (and of other spectrums) with respect to quasinilpotent equivalence.

THEOREM 4.1. *If $a \overset{q}{\sim} b$ then $sp(a, X) = sp(b, X)$.*

Proof. Of course, we have to show only that $r(a, X) \subseteq r(b, X)$. By Corollary 2.2 it is sufficient to prove that $H^i(\mathcal{O}_{z_0}(X), \beta) = 0$, $0 \leqslant i \leqslant n$, for every $z_0 \in r(a, X)$. Let ψ be a form having analytic coefficients in a neighborhood of z_0, satisfying $\beta(\zeta)\,\psi(\zeta) \equiv 0$. We may suppose that this neighborhood is a polydisc D and $D \subset r(a, X)$. Define

$$\psi^*(z) = \sum_k (-1)^{|k|} (a\backslash b)^{[k]} \frac{\partial^k \psi(z)}{k!}\,;$$

this series converges normally in D [8, Chap. 2], therefore ψ^* is a form having analytic coefficients on D, of the same degree as ψ. We shall prove that $\alpha(z)\,\psi^*(z) \equiv 0$, whence it will follow (since $z_0 \in r(a, X)$ and D is a polydisc, $D \subset r(a, X)$) that there exists another analytic form φ on D such that $\psi^*(z) \equiv \alpha(z)\,\varphi(z)$. Performing over ψ^* and φ the same transformation as over ψ but with a and b interchanged we shall obtain $\psi(z) \equiv \beta(z)\,\varphi^*(z)$, $z \in D$, and the proof will be completed. Therefore it remains to verify that

$$\alpha(z)\,\psi^*(z) \equiv 0, \qquad \sum_k (-1)^{|k|} (b\backslash a)^k \frac{\partial^k \psi^*(z)}{k!} \equiv \psi(z), \qquad z \in D,$$

and $\beta(z)\,\varphi^*(z) \equiv \psi(z)$, where

$$\varphi^*(z) = \sum_k (-1)^{|k|} (b\backslash a)^k \frac{\partial^k \varphi(z)}{k!}\,.$$

We have

$$\begin{aligned}
\alpha(z)\,\psi^*(z) &= \alpha(z) \sum_k (-1)^{|k|} (a\backslash b)^{[k]} \frac{\partial^k \psi(z)}{k!} \\
&= \sum_k (-1)^{|k|} (a\backslash b)^{[k]}\, \beta(z) \frac{\partial^k \psi(z)}{k!} \\
&\quad - \sum_k (-1)^{|k|} \sum_{i=1}^{n} (a\backslash b)^{[k+1_i]} \cdot s_i \wedge \frac{\partial^k \psi(z)}{k!}
\end{aligned}$$

$$= \sum_k (-1)^{|k|} (a\backslash b)^{[k]} \beta(z) \frac{\partial^k \psi(z)}{k!}$$

$$+ \sum_k (-1)^{|k|+1} \sum_{i=1}^{n} (a\backslash b)^{[k+1_i]} s_i \wedge \frac{\partial^k \psi(z)}{k!}$$

$$= \beta(z)\, \psi(z) + \sum_k{}' (-1)^{|k|} (a\backslash b)^{[k]}$$

$$\cdot \left[\beta(z) \frac{\partial^k \psi(z)}{k!} + \sum_{i=1, k_i \neq 0}^{n} s_i \wedge \frac{\partial^{k-1_i} \psi(z)}{(k-1_i)!} \right];$$

but $\beta(z)\, \psi(z) \equiv 0$ and one may easily verify that

$$\beta(z) \frac{\partial^k \psi(z)}{k!} + \sum_{i=1, k_i \neq 0}^{n} s_i \wedge \frac{\partial^{k-1_i} \psi(z)}{(k-1_i)!} \equiv 0$$

for every $k \neq 0$; indeed, introducing the Taylor expansion of ψ, $\psi(\zeta) = \sum_k (\zeta - z)^k\, \partial^k \psi(z)/k!$, in the identity $\beta(\zeta)\, \psi(\zeta) \equiv 0$, it follows that

$$0 \equiv \beta(\zeta)\, \psi(\zeta) \equiv [(\zeta_1 - b_1)\, s_1 + \cdots + (\zeta_n - b_n)\, s_n]\, \psi(\zeta)$$

$$\equiv [(\zeta_1 - z_1)\, s_1 + \cdots + (\zeta_n - z_n)\, s_n]\, \psi(\zeta)$$

$$+ [(z_1 - b_1)\, s_1 + \cdots + (z_n - b_n)\, s_n]\, \psi(\zeta)$$

$$\equiv \beta(z)\, \psi(\zeta) + \sum_{i=1}^{n} (\zeta_i - z_i)\, s_i \psi(\zeta)$$

$$\equiv \sum_k (\zeta - z)^k \beta(z) \frac{\partial^k \psi(z)}{k!} + \sum_k \sum_{i=1}^{n} (\zeta - z)^{k+1_i} s_i \wedge \frac{\partial^k \psi(z)}{k!}$$

$$\equiv \sum_k (\zeta - z)^k \beta(z) \frac{\partial^k \psi(z)}{k!} + \sum_k{}' (\zeta - z)^k \sum_{i=1, k_i \neq 0}^{n} s_i \wedge \frac{\partial^{k-1_i} \psi(z)}{(k-1_i)!}$$

$$\equiv \beta(z)\, \psi(z) + \sum_k{}' (\zeta - z)^k \left[\beta(z) \frac{\partial^k \psi(z)}{k!} + \sum_{i=1, k_i \neq 0}^{n} s_i \wedge \frac{\partial^{k-1_i} \psi(z)}{(k-1_i)!} \right],$$

whence

$$\beta(z) \frac{\partial^k \psi(z)}{k!} + \sum_{i=1, k_i \neq 0}^{n} s_i \wedge \frac{\partial^{k-1_i} \psi(z)}{(k-1_i)!} \equiv 0 \qquad \text{for} \quad k \neq 0.$$

Thus the first identity is verified. In order to verify the second identity, we shall take into account that, by definition of $\psi^*(z)$, we have

$$\begin{aligned}\frac{\partial^p\psi^*(z)}{p!} &= \sum_q (-1)^{|q|}(a\backslash b)^{[q]}\,\frac{\partial^{p+q}\psi(z)}{p!\,q!}\\ &= \sum_q (-1)^{|q|}\binom{p+q}{q}(a\backslash b)^{[q]}\,\frac{\partial^{p+q}\psi(z)}{(p+q)!},\end{aligned}$$

therefore

$$\begin{aligned}\sum_p (-1)^{|p|}(b\backslash a)^{[p]}\,\frac{\partial^p\psi^*(z)}{p!} &= \sum_p (-1)^{|p|}(b\backslash a)^{[p]}\\ &\quad\cdot\sum_q (-1)^{|q|}\binom{p+q}{q}(a\backslash b)^{[q]}\,\frac{\partial^{p+q}\psi(z)}{(p+q)!}\\ &= \sum_p\sum_q (-1)^{|p+q|}\binom{p+q}{q}(b\backslash a)^{[p]}(a\backslash b)^{[q]}\,\frac{\partial^{p+q}\psi(z)}{(p+q)!}\\ &= \sum_k (-1)^{|k|}\sum_{p\leqslant k}\binom{k}{p}(b\backslash a)^{[p]}(a\backslash b)^{[k-p]}\,\frac{\partial^k\psi(z)}{k!}\\ &= \psi(z) + \sum_k{}' \cdots = \psi(z),\end{aligned}$$

since

$$\sum_{p\leqslant k}\binom{k}{p}(b\backslash a)^{[p]}(a\backslash b)^{[k-p]} = (b\backslash b)^{[k]} = (b-b)^k = 0.$$

Finally, the verification of the third identity is very similar to the first.

The following remark (whose justification is exactly contained in the preceding proof) is similar to Theorem 2.3 from [3, Chap. 1].

Remark 4.4. If $H^i(\mathscr{U}(D, X), \alpha) = 0$, $0 \leqslant i \leqslant n-1$, for every open polydisc D, and $a \sim^q b$, then $H^i(\mathscr{U}(D, X), \beta) = 0$, $0 \leqslant i \leqslant n-1$.

Before stating the following theorem, it is necessary to recall some notions. The left resolvent set in $L(X)$ for the n-tuple $a = (a_1, ..., a_n)$, $\rho^l(a, L(X))$, is the set of all $z \in \mathbb{C}^n$ for which there exist $u_1, ..., u_n \in L(X)$ such that $\sum_{j=1}^n u_j(z_j - a_j) = id$. The left spectrum in $L(X)$ for the n-tuple a is the complement $\sigma^l(a, L(X)) = \mathbb{C}^n\backslash\rho^l(a, L(X))$. The right resolvent set, respectively the right spectrum are similary defined [1, Definition 1.3]. It is easy to verify that if $z \in \rho^l(a, L(X))$, then there exists an open neighborhood V of z and n $L(X)$-valued analytic functions $u_1, \dots, u_n$ on V such that $\sum_{j=1}^{n} u_j(\zeta)(\zeta_j - a_j) = id$, $\zeta \in V$, and a similar property is also valid for $\rho^r(a, L(X))$ [1, Lemma 1.4].

THEOREM 4.2. *If* $\lim_{|k| \to \infty} \|(a\backslash b)^{[k]}\|^{1/|k|} = 0$, *then* $\sigma^r(a, L(X)) \subseteq \sigma^r(b, L(X))$ *and* $\sigma^l(b, L(X)) \subseteq \sigma^l(a, L(X))$.

Proof. If $z \in \rho^l(a, L(X))$ then there exist an open polydisc D having the center in z, and n $L(X)$-valued analytic functions $u_1, \dots, u_n$ on D such that $\sum_{j=1}^{n} u_j(\zeta)(\zeta_j - a_j) \equiv id$, $\zeta \in D$. Let $u_j(\zeta) = \sum_k (\zeta - z)^k u_{jk}$ be the Taylor expansion of the function u_j on D, $1 \leqslant j \leqslant n$. We remark that for any j, the series $\sum_k u_{jk}(a\backslash b)^{[k]}$ is absolutely convergent. Denoting $v_j = \sum_k u_{jk}(a\backslash b)^{[k]}$ we have $\sum_{j=1}^{n} v_j(z_j - b_j) = id$, therefore $z \in \rho^l(b, L(X))$. Thus the proof is completed.

COROLLARY 4.1. *If* $a \overset{q}{\sim} b$, *then* $\sigma^l(a, L(X)) = \sigma^l(b, L(X))$ *and* $\sigma^r(a, L(X)) = \sigma^r(b, L(X))$.

THEOREM 4.3. *If* $\lim_{|k| \to \infty} \|(a\backslash b)^{[k]}\|^{1/|k|} = 0$, *then for any* $x \in X$ *we have* $\sigma(a, x) \subseteq \sigma(b, x)$.

Proof. If $z_0 \in \rho(b, x)$, then there exist an open polydisc D centered in z_0 and n X-valued analytic functions $\varphi_1, \dots, \varphi_n$ on D, such that $\sum_{j=1}^{n} (\zeta_j - b_j)\, \varphi_j(\zeta) \equiv x$, $\zeta \in D$. Denoting

$$\varphi_j{}^*(z) = \sum_k (-1)^{|k|}(a\backslash b)^{[k]} \frac{\partial^k \varphi_j(z)}{k!}, \qquad 1 \leqslant j \leqslant n,$$

one verifies in the same manner as in the proof of Theorem 4.1 that $\sum_{j=1}^{n} (z_j - a_j)\, \varphi_j{}^*(z) \equiv x$, $z \in D$; consequently, $z_0 \in \rho(a, x)$ and the proof is completed.

COROLLARY 4.2. *If* $a \overset{q}{\sim} b$, *then* $\sigma(a, x) = \sigma(b, x)$ *for any* $x \in X$.

COROLLARY 4.3. *If a and b are decomposable n-tuples and* $a \overset{q}{\sim} b$, *then* $sp(a, x) = sp(b, x)$ *for any* $x \in X$.

COROLLARY 4.4. *If a and b are decomposable n-tuples having the spectral capacities* $\mathscr{E}_a$ *and* $\mathscr{E}_b$ *and if* $a \overset{q}{\sim} b$, *then* $\mathscr{E}_a(F) = \mathscr{E}_b(F)$ *for any* $F \in \mathscr{F}(\mathbb{C}^n)$.

PROPOSITION 4.1. *If* $a \overset{q}{\sim} b$ *and a is decomposable, then b is also decomposable.*

Proof. This proposition is similar to Theorem 2.1 of [3, Chap. 2]. If $\mathscr{E}_a$ denotes the spectral capacity of a, we shall prove that $\mathscr{E}_a$ is a spectral capacity for b. We have only to verify that $\mathscr{E}_a(F)$ is invariant with respect to b and $sp(b, \mathscr{E}_a(F)) \subseteq F$. By applying Theorem 3.2 and Corollaries 3.4 and 4.2, we have $\mathscr{E}_a(F) = X_a(F) = X_b(F)$, hence $\mathscr{E}_a(F)$ is invariant with respect to b. Moreover, since $a \overset{q}{\sim} b$, it is obvious that $a/\mathscr{E}_a(F) \overset{q}{\sim} b/\mathscr{E}_a(F)$, therefore Theorem 4.1 gives us $sp(b, \mathscr{E}_a(F)) = sp(a, \mathscr{E}_a(F)) \subseteq F$. This completes the proof.

THEOREM 4.3. *Let a and b be two decomposable n-tuples on the Banach spaces X respectively Y, and let $u \in L(X, Y)$. Then the following two statements are equivalent.*

(i) $u\mathscr{E}_b(F) \subseteq \mathscr{E}_a(F)$ *for any* $F \in \mathscr{F}(\mathbb{C}^n)$.

(ii) $\lim_{|k|\to\infty} \| c^k(a, b)u \|^{1/|k|} = 0$.

The theorem is similar to Theorem 3.3 of [3, Chap. 2]. Let us explain the new notations. If $a \in L(X)$ and $b \in L(Y)$, then $c(a, b) = l(a) - r(b)$ is well defined as (linear continuous) operator on $L(X, Y)$ (see [3, Chap. 2, Sect. 3]); this allows us to define, $c(a, b) = (c(a_1, b_1), \ldots, c(a_n, b_n))$ for two commutative n-tuples a on X and b on Y; this is a commutative n-tuple of (linear continuous) operators on $L(X, Y)$, therefore we can define $c^k(a, b)$.

Roughly speaking, the proof of this theorem is similar to that of the quoted theorem and it is not difficult to complete the details.

COROLLARY 4.5. *If a is decomposable n-tuple and $u \in L(X)$, then the following two statements are equivalent.*

(i) $u\mathscr{E}_a(F) \subseteq \mathscr{E}_a(F)$ *for any* $F \in \mathscr{F}(\mathbb{C}^n)$.

(ii) $\lim_{|k|\to\infty} \| c^k(a, a)u \|^{1/|k|} = 0$.

COROLLARY 4.6. *If a and b are two decomposable n-tuples, then the following two statements are equivalent.*

(i) $\mathscr{E}_a(F) = \mathscr{E}_b(F)$ *for any* $F \in \mathscr{F}(\mathbb{C}^n)$.

(ii) $a \overset{q}{\sim} b$.

We shall close this section with a few results concerning two special classes of decomposable n-tuples, namely the spectral n-tuples (in Dunford's sense) and the scalar generalized n-tuples. The definitions are similar to those of the case $n = 1$ [5; 6; and 3, p. 94].

If a is a spectral n-tuple and E denotes a spectral measure for a, then $\mathscr{E}(F) = E(F)X$, $F \in \mathscr{F}(\mathbb{C}^n)$ is a spectral capacity for a. Then

Theorem 3.2 implies the uniqueness of the spectral measure for spectral n-tuples.

PROPOSITION 4.2. *If a and b are spectral n-tuples and E_a, E_b denotes their spectral measures, then $a \sim^q b$ iff $E_a = E_b$.*

Proof. See [3, Corollary 2.4].

PROPOSITION 4.3. *If a and b are spectral n-tuples, then*

$$\lim_{|k|\to\infty} \|(a\backslash b)^{[k]}\|^{1/|k|} = 0$$

implies

$$\lim_{|k|\to\infty} \|(b\backslash a)^{[k]}\|^{1/|k|} = 0.$$

Proof. See [3, Corollary 3.6].

PROPOSITION 4.4. *If a is a spectral n-tuple and $b \sim^q a$, then b is also a spectral n-tuple.*

Proof. See [3, Corollary 2.5]. The scalar generalized n-tuples has been studied (in a more generalized framework) by Albrecht in [1]; he uses the technique of tensor products. In order to show that the scalar generalized n-tuples are decomposables, we need the so-called Support Theorem [1, Theorem 2.9 and Corollaries 2.10, 2.12]: For every scalar generalized n-tuple (i.e. there exists a multiplicative $L(X)$-valued distribution u: $C^\infty(\mathbb{C}^n) \to L(X)$ such that $a_i = u(z_i)$, $1 \leqslant i \leqslant n$ and $u(1) = id$), we have $\operatorname{supp}(u) = sp(a, X)$. By applying this theorem one may verify that $\mathscr{E}(F) = \{x \in X, \operatorname{supp}(u(\cdot)x) \subseteq F\}$, is a spectral capacity for a, where we have denoted $\operatorname{supp}(u(\cdot)x)$ the support of the distribution $\varphi \to u(\varphi)x$, $\varphi \in C^\infty(\mathbb{C}^n)$.

The following theorem is similar to Theorem 1.5 of [3, Chap. 4].

THEOREM 4.3. *Let u: $C^\infty(\mathbb{C}^n) \to L(X)$ and v: $C^\infty(\mathbb{C}^n) \to L(Y)$ be two spectral distributions and let $a \in L(X, Y)$ be a linear continuous mapping such that $c(v(z_i), u(z_i))a = 0$, $1 \leqslant i \leqslant n$. Then there exists an integer $m \geqslant 0$ such that*

$$c(v(f_1), u(f_1)) \cdots c(v(f_{m+1}), u(f_{m+1}))\, a = 0 \quad \textit{for every } f_1, \ldots, f_{m+1} \in C^\infty(\mathbb{C}^n).$$

Proof. Consider the $L(X, Y)$-valued analytic function

$$\mathbb{C}^n \ni \xi \to \exp\left[\sum_{i=1}^{n} \xi_i v(z_i)\right] a \exp\left[-\sum_{i=1}^{n} \xi_i u(\bar{z}_i)\right];$$

as in the proof of the quoted theorem one may show that this function satisfies an inequality of the following form.

$$\left\| \exp\left[\sum_{i=1}^{n} \xi_i v(z_i)\right] a \exp\left[-\sum_{i=1}^{n} \xi_i u(\bar{z}_i)\right]\right\| \leqslant M \mid \xi \mid^k$$

(where $k = (k_1, ..., k_n)$ is an integer n-tuple and $| \xi |^k = | \xi_1 |^{k_1} \cdots | \xi_n |^{k_n}$). By applying Cauchy's inequalities [8, Theorem 2.2.7] it is seen that this function is a polynomial p of the degree at most $| k |$. Consequently,

$$\exp\left[\sum_{i=1}^{n} \xi_i v(z_i)\right] a \exp\left[-\sum_{i=1}^{n} \xi_i u(\bar{z}_i)\right] = p(\xi),$$

whence

$$p(\xi) \exp\left[\sum_{i=1}^{n} \xi_i u(\bar{z}_i)\right] = \exp\left[\sum_{i=1}^{n} \xi_i v(z_i)\right] a.$$

Let us remark that if a satisfies this equality with the polynomial p, then $c(v(\bar{z}_j), u(\bar{z}_j))a$ satisfies the same equality with the polynomial $\partial p / \partial \xi_j$, $1 \leqslant j \leqslant n$. Moreover, $c(v(z_i), u(z_i))\ c(v(\bar{z}_j), u(\bar{z}_j))a = 0$, $1 \leqslant i \leqslant n$. This remark allows us to obtain our statement inductively (as in [3]) with respect to the (total) degree of p.

COROLLARY 4.7. *If the n-tuple a has two spectral distributions u and v, then there exists an integer $m \geqslant 0$ such that $(u(f)\backslash v(f))^{[m+1]} = 0$ for every $f \in C^\infty(\mathbb{C}^n)$.*

REFERENCES

1. E. ALBRECHT, Functionalkalküle in mehreren veränderlichen, Ph.D. Dissertation, Mainz University, 1972.
2. C. APOSTOL, Spectral decomposition and functional calculus, *Rev. Roumaine Math. Pures Appl.* **10** (1968), 1481–1528.
3. I. COLOJOARĂ AND C. FOIAŞ, "Theory of Generalized Spectral Operators," Gordon and Breach, New York/London/Paris, 1968.
4. I. COLOJOARĂ AND C. FOIAŞ, The Riesz–Dunford calculus with decomposable operators, *Rev. Roumaine Math. Pures Appl.* **5** (1967), 627–641.
5. N. DUNFORD, Spectral operators, *Pacific J. Math.* **4** (1954), 321–354.
6. N. DUNFORD, A survey of the theory of spectral operators, *Bull. Amer. Math. Soc.* **64** (1958), 217–274.
7. C. FOIAŞ, Spectral capacities and decomposable operators, *Rev. Roumaine Math. Pures Appl.* **10** (1968), 1539–1545.

8. L. Hörmander, "An Introduction to Complex Analysis in Several Variables," Van Nostrand, Toronto/New York/London, 1966.
9. J. L. Taylor, A joint spectrum for several commuting operators, *J. Functional Analysis* **6** (1970), 172–191.
10. J. L. Taylor, Analytic functional calculus for several commuting operators, *Acta Math.* **125** (1970), 1–38.